

On the impact of the penalty on the cancellable American options

Tsvetelin Zaeovski^{a,b}

^a*Institute of Mathematics and Informatics, Bulgarian Academy of Sciences,
Acad. Georgi Bonchev Str., Block 8, 1113 Sofia, Bulgaria*

^b*Faculty of Mathematics and Informatics, Sofia University “St. Kliment Ohridski”,
5, James Bourchier Blvd., Sofia 1164, Bulgaria*

t_s_zaevski@math.bas.bg, t_s_zaevski@abv.bg (T. Zaeovski)

Received: 13 January 2025, Revised: 4 October 2025, Accepted: 4 October 2025,
Published online: 21 October 2025

Abstract The cancellable American options, also known as game options, are financial instruments that give a canceling right to the option’s writer in addition to the existing such holder’s right. The writer owes some penalty above the usual option payoff for using this right. We assume that this penalty consists of three parts – a proportion of the usual payoff, some number of shares of the underlying asset, and a fixed amount. It turns out that a cancellable option can be of one of the following three types – a regular American option, an American-style derivative that expires either at the maturity or when the underlying asset reaches the strike, or a real cancellable option. In this paper, the impact of the penalty on the option’s type is investigated. The perpetual case is only explored having in mind that it determines the kind of the finite maturity instruments in some sense.

Keywords Cancellable American options, game options, optimal boundaries, optimal strategies, impact of the penalty

2020 MSC 42A38, 60G40, 60J65

1 Introduction

The financial derivatives are one of the most important financial instruments against the market risk. By their very nature, they are a major indicator of the investor expectations for the future market behavior. On the other hand, the options are one of the most traded derivatives. They preserve their holder from the market fluctuations

© 2026 The Author(s). Published by VTeX. Open access article under the [CC BY](#) license.

when he wants to buy (calls) or sell (puts) some asset at a price no larger (calls) or lower (puts) than a predefined level known as the strike price or simply the strike. Thus the options can be viewed as an insurance instrument. Furthermore, the price structure they generate w.r.t. the strike is very informative for the investors beliefs. On the other hand, two main types – European and American – can be distinguished depending on when the contracts expire. For the European style options, the exercise can be done only at a predefined maturity date. Alternatively, the American options give to their owner the right to chose the moment for exercising. There are further modifications known as exotic options – barrier, Asian, look-back, digital, straddle, strangle, and many others. Although the options exhibit such variety, the American ones have a largest segment amongst all traded options namely due to the property of early exercising preferred by investors. However, there is some difference between the holder and the writer of these options since the writer has only obligations. To overcome this, a new class of financial instruments has been designed, known as the cancellable American options. The main feature distinguishing these derivatives from the regular American options is the writer's right to cancel the contract prematurely paying some penalty above the usual payoff – a traditional assumption is that it is fixed during the option's life, but we shall examine more complicated cases.

These derivatives are first introduced in the scientific literature by [Kifer \(2000\)](#) under the name game options. Later, the term *Israeli* is also used, see [Kifer \(2013\)](#). Regardless of the penalty structure, these financial instruments fall in the field of the so-called Dynkin stochastic games (see [Dynkin \(1969\)](#)). Thus their pricing problem turns into finding of the optimal strategies (if they exist) for both of option's writer and holder. In the stochastic terms, these strategies have to form a saddle point in the field of the stopping times w.r.t. the natural filtration. Some existence results for the models based on diffusion processes can be found in [Friedman \(1973\)](#), [Bensoussan and Friedman \(1974\)](#), [Bensoussan and Friedman \(1977\)](#), [Ekström \(2006\)](#), [Karatzas and Sudderth \(2006\)](#), [Ekström and Villeneuve \(2006\)](#), [Gapeev and Lerche \(2011\)](#). The results for significantly larger classes of process are derived in [Ekström and Peskir \(2008\)](#) and [Peskir \(2009\)](#) – it turns out that the right continuity leads to the Stackelberg equilibrium, whereas a left continuity w.r.t. the stopping times (quasi-left continuity) is necessary for the Nash equilibrium. Note that some of the most applied stochastic processes in financial modeling exhibit both of these requirements, for example, the Lévy processes, the stochastic differential equations they generate, particularly the diffusions, etc.

Several important works devoted to the game options are published after [Kifer \(2000\)](#). The call style instruments are examined first in [Kunita and Seko \(2004\)](#). These results are refined later by [Emmerling \(2012\)](#) and [Yam et al. \(2014\)](#). The put style options are explored in [Kyprianou \(2004\)](#), [Ekström \(2006\)](#), [Suzuki and Sawaki \(2007\)](#), and [Kühn and Kyprianou \(2007\)](#). Some exotic game options are investigated in [Kyprianou \(2004\)](#) (Russian), [Baurdoux and Kyprianou \(2004\)](#) (integral options, related to the Asian ones), [Gapeev \(2005\)](#) (spread options), [Ekström \(2006\)](#) (capped options), [Guo et al. \(2014\)](#) (look-back), [Guo et al. \(2020\)](#) (Asian). The cancellable options under some generalized assumptions are examined in [Kallsen and Kühn \(2004\)](#), [Hamadène \(2006\)](#), [Kühn et al. \(2007\)](#), [Dumitrescu et al. \(2017\)](#), [Guo and Rutkowski \(2017\)](#), [Guo \(2020\)](#), [Dolinsky \(2020\)](#), and [Palmowski and Stępiak \(2023\)](#).

We position the present study in the framework by [Black and Scholes \(1973\)](#) – the underlying asset is driven by a log-normal process. In addition to the usual assumption

for a fixed penalty, we consider a three-component structure – a proportion of the usual payoff, shares of the underlying asset, and a fixed amount. These instruments are investigated with and without maturity constraints in [Zaevski \(2023, 2025b\)](#), see also [Ekström and Villeneuve \(2006\)](#) and [Zaevski \(2020a\)](#) for cancellable options with proportional penalties. Regardless of the penalty structure, it turns out that these instruments may exhibit three different behaviors. In all of them, the holder's exercise set contains all points below/above some boundary for the put/call style options. The distinction comes from the shape of the writer's optimal set. For some large enough penalties, the premature cancellation is never optimal – the canceling price is larger than the expected losses. In such a way, the option is regular American. On the other hand, the relatively middle values make the first hitting to the strike the unique writer's optimal strategy. In this case, the option is rather American than game, since it can be viewed as a derivative that gives an early exercise right to its holder and expires at the maturity or when the underlying asset hits the strike – in the last case the holder receives a predefined amount. If it is L , then we shall entitle the option L -American. Finally, the canceling right has a real impact for the small enough penalties – we shall use the name *real game* or *real cancellable* in this case. The writer's optimal set is an interval with the strike for the right/left endpoint for the puts/calls. The options without maturity horizon determine which of these three cases holds in the sense that they contain the whole information. First, if the perpetual option is regular American, then all finite maturity options are regular American too. Second, if the option is of the L -American kind, then there exists a critical value for the time to maturity above which the option is L -American too, but it is regular American for the lower maturities. Third, if the perpetual option is real cancellable, then there exist two critical values for the time to maturity where the finite maturity option changes its behavior. This importance of the perpetual case motivates the present research. For a survey of such instruments, we refer to [Kyprianou \(2004\)](#), [Kunita and Seko \(2004\)](#), [Ekström \(2006\)](#), [Suzuki and Sawaki \(2007\)](#), [Emmerling \(2012\)](#), [Yam et al. \(2014\)](#), [Zaevski \(2020a,b,c, 2023\)](#), and [Gapeev et al. \(2021\)](#).

The main results of this paper are in the recognition which values of the penalty triple (proportion, shares, and fixed amount) to which case lead. We derive the critical values that distinguish the three possible types. It is interesting to be mentioned that the first component (the proportion of the usual payoff) does not influence the transition between the regular American and L -American type. On the other hand, if the payoff taken at the strike (it depends only on the penalty components related to the number of shares and the fixed amount) is less than the price of the at-the-money regular American option (the initial asset price is the strike), then the option is either L -American or real cancellable. Here appears the impact of the first penalty component. Next we derive iteratively all critical values in this order: proportion of the payoff, number of shares, and finally, the fixed amount. Based on these results, we provide an algorithm for recognizing the option's type. These relations are examined in detail for the put options, whereas the calls are examined through some symmetry arguments. Several numerical experiments are provided to illustrate and validate the theoretical findings.

It is worth to mention that along with this, we investigate the optimal sets of the L -American options as well as their pricing rules.

The paper is organized as follows. The base we use later is given in Section 2. The

L -American options are discussed in Section 3. The results for the put options are obtained in Section 4, whereas the calls are investigated in Section 5. Some numerical examples are provided in Section 6.

2 Preliminaries

Let the underlying asset be driven by the geometric Brownian motion

$$dS_t = rS_t dt + \sigma S_t dB_t \quad (1)$$

under the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$. We shall use a superscript to mark the initial value, i.e. $\mathbb{E}^{t,x}$ means the expectation under the assumption $S_t = x$. Also, if $t = 0$, we shall mark only the dependence on x . On the other hand, somewhere it is more appropriate if we mark the dependence directly in the process, i.e. S_t^x notates the process if $S_0 = x$. The measure $\mathbb{Q}$ is risk-neutral w.r.t. to the risk-free rate r . We assume that it is a constant during the option's life – note that it can take negative values. We introduce an extra discount factor with rate λ , assuming $\lambda \geq 0$ and $r + \lambda > 0$. Let the option's strike price be denoted by K . The holder of a regular American option (put or call) receives the amount of

$$\begin{aligned} N_1(t, x) &= e^{-\lambda t} (K - x)^+, \quad \text{put,} \\ N_1(t, x) &= e^{-\lambda t} (x - K)^+, \quad \text{call,} \end{aligned} \quad (2)$$

if he exercises in a moment t at the spot price $S_t = x$. Additionally, the cancellable American option gives its writer the right to cancel prematurely paying some amount above the usual payoff. We assume that it consists of three parts: $\eta_1 \geq 1$ being proportion of the payoff, $\eta_2 \geq 0$ shares of the underlying asset, and a fixed amount of $\eta_3 \geq 0$. Thus the holder owes the total amount of

$$\begin{aligned} N_2(t, x) &= e^{-\lambda t} [\eta_1 (K - x)^+ + \eta_2 x + \eta_3], \quad \text{put,} \\ N_2(t, x) &= e^{-\lambda t} [\eta_1 (x - K)^+ + \eta_2 x + \eta_3], \quad \text{call,} \end{aligned} \quad (3)$$

if he cancels the option. As a consequence, we have a stochastic game (see [Dynkin \(1969\)](#)) between two players – the option's holder and writer.

Let us denote by $n_1(x)$ and $n_2(x)$ the respective undiscounted payoffs, i.e.

$$\begin{aligned} n_1(x) &= (K - x)^+, \\ n_2(x) &= \eta_1 (K - x)^+ + \eta_2 x + \eta_3 \end{aligned} \quad (4)$$

for the puts and

$$\begin{aligned} n_1(x) &= (x - K)^+, \\ n_2(x) &= \eta_1 (x - K)^+ + \eta_2 x + \eta_3 \end{aligned} \quad (5)$$

for the calls.

Remark 1. The discount factor λ can be viewed as a dividend rate due to Proposition 2.2 from [Zaeviski \(2025a\)](#). It says that if there are dividends payable at rate δ , then this model is equivalent to a nondividend one with parameters $\bar{r} = r - \delta$ and $\bar{\lambda} = \lambda + \delta$ in

the sense that both models lead to equal option prices. This parametrization, used in [McKean \(1965\)](#) and [Shiryaev et al. \(1995\)](#), allows some computational facilities. We shall refer to that change of parameters when the use of the classical parametrization with the dividend rate is necessary. We shall distinguish both approaches using the name *discount parametrization* when the model is defined by asset (1) and payoffs (2)–(3). Alternatively, we shall use the name *dividend parametrization* when the asset's drift is compensated by the dividend rate (from r to $r - \delta$), but the payoffs are undiscounted as in formulas (4)–(5).

We assume now that $t = 0$ since the model is time-homogeneous in the perpetual case. Suppose that the buyer's (holder's) strategy is to exercise in the stopping time τ^b and the seller's (writer's) one is another stopping time τ^s . The financial result of these strategies at the point x is

$$\begin{aligned} M(x; \tau^b, \tau^s) \\ = \mathbb{E}^x \left[e^{-r\tau^b} N_1(\tau^b, S_{\tau^b}) I_{\tau^b \leq \tau^s} + e^{-r\tau^s} N_2(\tau^s, S_{\tau^s}) I_{\tau^s < \tau^b} \right]. \end{aligned} \quad (6)$$

The option's holder/writer has to maximize/minimize the value of (6) w.r.t. all stopping times. Based on Theorem 2.1 from [Ekström and Peskir \(2008\)](#), it is proven in [Zaevski \(2023\)](#) that this stochastic game exhibits a Nash equilibrium (see also [Peskir \(2009\)](#)). We shall denote its value function by

$$V(x) = \inf_{\tau^s \in \mathcal{T}} \sup_{\tau^b \in \mathcal{T}} M(x; \tau^b, \tau^s) = \sup_{\tau^b} \inf_{\tau^s} M(x; \tau^b, \tau^s), \quad (7)$$

where $\mathcal{T}$ is the set of all stopping times. Hence, $n_1(x) \leq V(x) \leq n_2(x)$. The following lemma gives the time-relations for the price function.

Lemma 2.1. *If the price function at the point (t, x) is $\bar{V}(t, x)$, then $\bar{V}(t, x) = e^{-\lambda t} V(x)$.*

Proof. Let us denote by $\bar{M}(t, x; \tau^b, \tau^s)$ the financial result of the strategies $\tau^b \geq t$ and $\tau^s \geq t$ under the assumption $S_t = x$. The lemma holds since

$$\begin{aligned} \bar{M}(t, x; \tau^b, \tau^s) \\ = \mathbb{E}^{t, x} \left[e^{-r(\tau^b - t)} N_1(\tau^b, S_{\tau^b}) I_{\tau^b \leq \tau^s} + e^{-r(\tau^s - t)} N_2(\tau^s, S_{\tau^s}) I_{\tau^s < \tau^b} \right] \\ = e^{-\lambda t} \mathbb{E}^{t, x} \left[e^{-(r+\lambda)(\tau^b - t)} n_1(S_{\tau^b}) I_{\tau^b \leq \tau^s} + e^{-(r+\lambda)(\tau^s - t)} n_2(S_{\tau^s}) I_{\tau^s < \tau^b} \right]. \end{aligned} \quad (8)$$

□

Having in mind Lemma 2.1, we define the holder's and writer's optimal sets as the points (t, x) for which $V(x) = n_1(x)$ or $V(x) = n_2(x)$, respectively.¹ Thus the optimal strategies τ^b and τ^s can be defined as

$$\begin{aligned} \tau^b &= \inf \{t : V(S_t) = n_1(S_t)\}, \\ \tau^s &= \inf \{t : V(S_t) = n_2(S_t)\}. \end{aligned} \quad (9)$$

¹These conditions are equivalent to $\bar{V}(t, x) = N_1(t, x)$ or $\bar{V}(t, x) = N_2(t, x)$.

We shall denote the optimal sets by Y^b and Y^s . The boundaries of these sets are known as early exercise or optimal boundaries. We shall discuss in detail the particular form of the optimal sets and their boundaries in the corresponding sections devoted to the put and call options.

The existence of two optimal boundaries (holder's and writer's) leads to a problem for the first exit from a strip of the underlying asset or, equivalently, of a Brownian motion. We need the following well-known result for diffusion processes – for the proof see [Darling and Siebert \(1953\)](#) or [Lehoczky \(1977\)](#).

Lemma 2.2. *Let the diffusion process X_t be defined as*

$$dX_t = \mu(X_t) dt + \sigma(X_t) dB_t \quad (10)$$

for some Lipschitz functions $\mu(\cdot)$ and $\sigma(\cdot)$. Let the initial value be between a and b , $a < X_0 < b$. Let τ^a and τ^b be the first hitting moments of X_t to the values a and b , respectively. Let the pair $\{f_1(u), f_2(u)\}$ consist of any two fundamental (linearly independent) solutions of the ordinary differential equation (ODE)

$$\frac{1}{2}\sigma^2(u)f''(u) + \mu(u)f'(u) - yf(u) = 0. \quad (11)$$

Under these assumptions, the following relations hold for a positive constant y :

$$\begin{aligned} \mathbb{E} \left[e^{-y\tau^a} I_{\tau^a < \tau^b} \right] &= \frac{f_1(X_0)f_2(b) - f_1(b)f_2(X_0)}{f_1(a)f_2(b) - f_1(b)f_2(a)}, \\ \mathbb{E} \left[e^{-y\tau^b} I_{\tau^b < \tau^a} \right] &= \frac{f_1(a)f_2(X_0) - f_1(X_0)f_2(a)}{f_1(a)f_2(b) - f_1(b)f_2(a)}. \end{aligned} \quad (12)$$

If diffusion (10) is a Brownian motion with drift, then the functions $\mu(\cdot)$ and $\sigma(\cdot)$ are constants, and the last one is equal to one. Thus ODE (11) turns into

$$f''(u) + 2\mu f'(u) - 2yf(u) = 0. \quad (13)$$

It is characterized by the quadratic equation

$$u^2 + 2\mu u - 2y = 0 \quad (14)$$

the solutions of which are

$$u_{1,2} = -\mu \pm \sqrt{\mu^2 + 2y}. \quad (15)$$

Note that they are real since $y \geq 0$. Thus a pair of fundamental solutions is $f_{1,2}(u) = e^{u_{1,2}u}$. Therefore, the Laplace transforms (12) can be written as

$$\begin{aligned} \mathbb{E} \left[e^{-y\tau^a} I_{\tau^a < \tau^b} \right] &= e^{\mu(a-X_0)} \frac{e^{(b-X_0)\sqrt{\mu^2+2y}} - e^{-(b-X_0)\sqrt{\mu^2+2y}}}{e^{(b-a)\sqrt{\mu^2+2y}} - e^{-(b-a)\sqrt{\mu^2+2y}}}, \\ \mathbb{E} \left[e^{-y\tau^b} I_{\tau^b < \tau^a} \right] &= e^{\mu(b-X_0)} \frac{e^{(X_0-a)\sqrt{\mu^2+2y}} - e^{-(X_0-a)\sqrt{\mu^2+2y}}}{e^{(b-a)\sqrt{\mu^2+2y}} - e^{-(b-a)\sqrt{\mu^2+2y}}}. \end{aligned} \quad (16)$$

Suppose that the underlying asset starts from the point $S_0 = x$. Note that the first exit of process (1) from a strip $0 < A < S_0 < B$ is equivalent to the exit of a Brownian motion with drift

$$\mu = \frac{r}{\sigma} - \frac{\sigma}{2} \quad (17)$$

from the strip (a, b) , where $a = \frac{1}{\sigma} \ln \frac{A}{x}$ and $b = \frac{1}{\sigma} \ln \frac{B}{x}$. Thus the Laplace transforms (16) taken at the point of the total discount rate, $y = r + \lambda$, turn into

$$\begin{aligned}\mathbb{E} \left[e^{-(r+\lambda)\tau^a} I_{\tau^a < \tau^b} \right] &= \left(\frac{A}{x} \right)^q \frac{B^p - x^p}{B^p - A^p}, \\ \mathbb{E} \left[e^{-(r+\lambda)\tau^b} I_{\tau^b < \tau^a} \right] &= \left(\frac{B}{x} \right)^q \frac{x^p - A^p}{B^p - A^p},\end{aligned}\tag{18}$$

where the constants p and q are expressed in roots (15) as

$$\begin{aligned}p &= u_1 - u_2 = 2\sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r+\lambda}{\sigma^2}}, \\ q &= -u_2 = \sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r+\lambda}{\sigma^2}} + \left(\frac{r}{\sigma^2} - \frac{1}{2}\right).\end{aligned}\tag{19}$$

We have $p \geq q + 1$ and the equality holds only when $\lambda = 0$. We can derive the Laplace transforms of the one-sided hits taking $B = \infty$ and $A = 0$, respectively:

$$\begin{aligned}\mathbb{E} \left[e^{-(r+\lambda)\tau^a} I_{\tau^a < \infty} \right] &= \left(\frac{A}{x} \right)^q, \\ \mathbb{E} \left[e^{-(r+\lambda)\tau^b} I_{\tau^b < \infty} \right] &= \left(\frac{x}{B} \right)^{p-q}.\end{aligned}\tag{20}$$

We need the following relations between the constants p and q .

Lemma 2.3. $r < 0$ if and only if $p > 2q + 1$.

Lemma 2.4. The following statements related to the sign of r hold:

1. If $r < 0$, then

$$\frac{q+1}{p-q} < 1 < l,\tag{21}$$

where

$$l := \left(\frac{p-q-1}{q} \right)^{p-q-1} \left(\frac{q+1}{p-q} \right)^{p-q}.\tag{22}$$

2. If $r > 0$, then

$$\frac{q+1}{p-q} > l > 1.\tag{23}$$

3. If $r = 0$, then

$$l = \frac{q+1}{p-q} = 1.\tag{24}$$

Proof. The first part of inequality (21) follows from Lemma 2.3. If we consider the term (22) as a function of p , say $l(p)$, then its derivative is

$$l'(p) = l(p) \ln \left(\frac{(p-q-1)(q+1)}{q(p-q)} \right) > 0.\tag{25}$$

Therefore, $l(p)$ is an increasing function. Having in mind $p > 2q + 1$ (due to Lemma 2.3), we obtain $l(p) > l(2q + 1) = 1$.

Inequalities (23) can be proven in the same manner using the inequality $p < 2q + 1$ that holds when $r > 0$. Finally, if $r = 0$, then Lemma 2.3 leads to $p = 2q + 1$ which proves the third statement. $\square$

3 L-American options

Let us define a new American-style financial instrument – we name it an L -American option.

Definition 3.1. Let L be a positive constant. An L -American option with strike K expires when the underlying asset hits the strike paying amount of $e^{-\lambda\tau}L$, where τ is just this hitting moment. Furthermore, the holder may exercise the option at every moment t receiving the amount of $N_1(t, S_t)$.

We can view the L -American options as financial instruments with stochastic maturity (the moment when the asset reaches the strike) with final payout L . In this light, the payoff $N_1(t, x)$ that the holder can receive is continuous whereas the payout at this stochastic maturity is a specification of the option contract. Note that the points not below (not above) the strike are never optimal for the puts (calls) since the payoff is zero-valued in this region.

We shall denote the price of an L -American option by $V_L(\cdot)$. Thus, if we denote by $\tau^{y,x}$ the first hitting moment of an asset starting at y to the value x , then

$$V_L(x) = \sup_{\tau \in \mathcal{T}} \left\{ \begin{aligned} &\mathbb{E} \left[e^{-\lambda\tau} e^{-r\tau} (K - S_\tau^x)^+ I_{\tau < \tau^{x,K}} \right] \\ &+ L \mathbb{E} \left[e^{-\lambda\tau^{x,K}} e^{-r\tau^{x,K}} I_{\tau^{x,K} \leq \tau} \right] \end{aligned} \right\}. \quad (26)$$

Let $\mathcal{B}$ be the differential operator

$$(\mathcal{B}f) = rx f'(x) + \frac{\sigma^2}{2} x^2 f''(x) - (r + \lambda) f(x). \quad (27)$$

We need the following well-known result for the optimal stopping problems – see for example [van Moerbeke \(1973\)](#) or [Jacka \(1992\)](#).

Lemma 3.2. *If a point x is optimal, then $(\mathcal{B}n_1)(x) < 0$, where the function $n_1(\cdot)$ is given in (4) for the puts and in (5) for the calls. In the put case, this is equivalent to $\lambda x - (r + \lambda)K < 0$ when x is below the strike. The inverse inequality holds for the calls when x is above the strike.*

We shall prove a proposition that characterizes the optimal sets of L -American options.

Proposition 3.3. *If a point x is optimal for an L -American put option, then all points $0 \leq y < x$ are optimal too.*

Proof. First, note that $x < K$ since it is optimal. Suppose that the point y is not optimal. Therefore, there exists a stopping time τ such that

$$K - y < \mathbb{E} \left[e^{-(r+\lambda)\tau} (K - S_\tau^y)^+ I_{\tau < \tau^{y,K}} \right] + L \mathbb{E} \left[e^{-(r+\lambda)\tau^{y,K}} I_{\tau^{y,K} \leq \tau} \right]. \quad (28)$$

If $\bar{\tau} = \tau \wedge \tau^{y,x}$, then inequality (28) holds for $\bar{\tau}$ too, since the point x is optimal. Therefore, using the Dynkin formula, we derive

$$\begin{aligned} 0 &< \mathbb{E} \left[e^{-(r+\lambda)\bar{\tau}} (K - S_{\bar{\tau}}^y)^+ I_{\bar{\tau} < \tau^{y,K}} \right] + L \mathbb{E} \left[e^{-(r+\lambda)\tau^{y,K}} I_{\tau^{y,K} \leq \bar{\tau}} \right] - K + y \\ &= \mathbb{E} \left[e^{-(r+\lambda)\bar{\tau}} (K - S_{\bar{\tau}}^y) \right] - K + y \\ &= \mathbb{E} \left[\int_0^{\bar{\tau}} \lambda S_u^y - (r + \lambda) K du \right] < 0. \end{aligned} \quad (29)$$

The last inequality is true because Lemma 3.2 shows that $\lambda S_u - (r + \lambda) K < 0$ whenever $S_u < x$. The contradiction finishes the proof. $\square$

Proposition 3.3 means that the optimal set of an L -American put option contains all points below some boundary. Furthermore, this boundary is a constant during the time, since there are no maturity constraints. Under the dividend parametrization, the optimal boundary of a put style L -American option as well as its price are discussed in Theorem 2 of Kyprianou (2004), in Section 3.1 of Ekström (2006), and in Theorem 3.1 of Suzuki and Sawaki (2007); for the calls, see formulas (3.1) from Emmerling (2012) and (2.12) from Yam et al. (2014). We need the following lemma prior to providing the related results under the discount parametrization.

Lemma 3.4. *Let the function $h(a; \xi)$ be defined as*

$$h(a; \xi) = -a^{p+1} (p - q - 1) + a^p (p - q) - a^{p-q} p \xi - a (q + 1) + q. \quad (30)$$

Its behavior in the interval $(0, 1)$ is as follows: it starts from the positive value $h(0; \xi) = q$, decreases having a unique root after which stays always negative.

Proof. The proof can be found in Appendix B.2 of Zaeveski (2020c) (for $k = 1$). $\square$

Proposition 3.5. *For $\xi = \frac{L}{K}$, let $a^* \in (0, 1)$ be the unique root of function (30) in the interval $(0, 1)$. The holder's optimal boundary is $A^* = K a^*$. The price is given by*

$$V_L(x) = \begin{cases} K - x, & x \leq A^*, \\ (K - A^*) \left(\frac{A^*}{x}\right)^q \frac{K^p - x^p}{K^p - A^{*p}} + L \left(\frac{K}{x}\right)^q \frac{x^p - A^{*p}}{K^p - A^{*p}}, & x \in (A^*, K), \\ L \left(\frac{K}{x}\right)^q, & x \geq K. \end{cases} \quad (31)$$

Proof. As we mentioned above, Proposition 3.3 shows that the optimal points are below some flat boundary. Let the initial asset value x be large enough but below the strike. Supposing that the optimal boundary is A and using formulas (18), we obtain the price as a function of A as

$$\tilde{V}_L(A) = (K - A) \left(\frac{A}{x}\right)^q \frac{K^p - x^p}{K^p - A^p} + L \left(\frac{K}{x}\right)^q \frac{x^p - A^p}{K^p - A^p}. \quad (32)$$

Normalizing by $\xi = \frac{L}{K}$, $y = \frac{x}{K}$, and $a = \frac{A}{K}$, we transform price (32) into

$$\tilde{V}_L(a) = \frac{K}{y^q} \frac{(1 - a) a^q (1 - y^p) + \xi (y^p - a^p)}{1 - a^p}. \quad (33)$$

Its derivative is

$$\widetilde{V}'_L(a) = \frac{K}{y^q} \frac{1 - y^p}{(1 - a^p)^2} a^{q-1} h(a; \xi). \quad (34)$$

Lemma 3.4 shows that function (30) has a unique root in the interval $(0, 1)$. Furthermore, it leads to the maximum of the price function. Once we derive the optimal boundary A^* , we obtain the prices in (31) through formulas (18) and (20). $\square$

Some symmetrical arguments lead to the result for the call style L -American options.

Proposition 3.6. *If $\lambda = 0$, then the early exercise is never optimal for the holder of an L -American call. Its price is*

$$V_L(x) = \begin{cases} L \left(\frac{x}{K} \right)^{p-q}, & x \leq K, \\ L \left(\frac{K}{x} \right)^q, & x > K. \end{cases} \quad (35)$$

If $\lambda > 0$, then the holder's optimal set consists of all points above some boundary A^ . It can be presented as $A^* = Ka^*$, where a^* is the unique root, larger than one, of function (30) taken for $\xi = -\frac{L}{K}$. The option price is given by*

$$V_L(x) = \begin{cases} L \left(\frac{x}{K} \right)^{p-q}, & x \leq K, \\ L \left(\frac{K}{x} \right)^q \frac{A^{*p} - x^p}{A^{*p} - K^p} + (A^* - K) \left(\frac{A^*}{x} \right)^q \frac{x^p - K^p}{A^{*p} - K^p}, & x \in (K, A^*), \\ x - K, & x \geq A^*. \end{cases} \quad (36)$$

Proof. We shall consider only the case $\lambda = 0$. We can proceed analogously to Proposition 3.5 when $\lambda > 0$. Note that if $\lambda = 0$, then $r > 0$ since $r + \lambda > 0$. Also, functions $N_1(\cdot)$ and $n_1(\cdot)$ coincide in this case. Suppose that a point x is optimal for the holder. Obviously $x > K$. Using the martingality of $e^{-rt} S_t$, we obtain for a finite stopping time ζ :

$$\begin{aligned} & E \left[e^{-r\zeta} n_1 \left(S_\zeta^x \right) I_{\zeta \leq \tau^{x,K}} + e^{-r\tau^{x,K}} L I_{\tau^{x,K} < \zeta} \right] \\ & \leq n_1(x) = x - K \\ & = E \left[e^{-r(\zeta \wedge \tau^{x,K})} S_{\zeta \wedge \tau^{x,K}}^x \right] - K \\ & = E \left[e^{-r\zeta} S_\zeta^x I_{\zeta \leq \tau^{x,K}} \right] + E \left[e^{-r\tau^{x,K}} S_{\tau^{x,K}}^x I_{\tau^{x,K} < \zeta} \right] - K \\ & < E \left[e^{-r\zeta} \left(S_\zeta^x - K \right) I_{\zeta \leq \tau^{x,K}} + e^{-r\tau^{x,K}} \left(S_{\tau^{x,K}}^x - K \right) I_{\tau^{x,K} < \zeta} \right] \\ & = E \left[e^{-r\zeta} \left(S_\zeta^x - K \right) I_{\zeta \leq \tau^{x,K}} \right] \\ & \leq E \left[e^{-r\zeta} n_1 \left(S_\zeta^x \right) I_{\zeta \leq \tau^{x,K}} + e^{-r\tau^{x,K}} L I_{\tau^{x,K} < \zeta} \right]. \end{aligned} \quad (37)$$

The contradiction finishes the proof. $\square$

Remark 2. Note that the holder's stopping region can be empty for a call L -American option (when $\lambda = 0$) whereas this is impossible for the puts. The difference comes from the fact that function (30) taken for $\xi = \frac{L}{K}$ always has a root in the interval $(0, 1)$, whereas if it is taken for $\xi = -\frac{L}{K}$, then the larger than one root exists only when $\lambda > 0$.

The absence of roots when $\lambda = 0$ can be interpreted as an infinitely large holder's optimal boundary.

We need the following definition for further distinction.

Definition 3.7. We shall say that the option is real cancellable if the writer's optimal region is neither the empty set nor the singleton $\{K\}$.

Remark 3. We have $n_1(x) = n_2(x)$ for the put-styled options only when $\{\eta_2 = 0, \eta_3 = 0, x \geq K\}$. In this case, the respective points are both optimal for the writer and holder due to the mathematical definition. To avoid the embarrassing circumstance that the holder would exercise without receiving anything, we shall exclude these points from the holder's optimal set. This may lead to an open holder's optimal set. This case is studied in [Ekström \(2006\)](#) and [Zaevski \(2020a\)](#). The optimal sets are $\Upsilon^s = [K, \infty)$ and $\Upsilon^b = (0, K)$ when $r \geq 0$. On the other hand, the option can be viewed as L -American since the immediate exercise and the first hit to the strike give a zero-result for the writer when the initial asset price is above the strike. Otherwise, if $r < 0$, then we have a real cancellable option. Some symmetrical arguments lead to analogous results for the calls when $\{\eta_2 = 0, \eta_3 = 0, x \leq K\}$. We shall exclude these cases hereafter.

4 Put options

For our further purposes, we need the price of the regular American options under the perpetual assumptions. The optimal boundary and the price can be obtained in a closed form since the boundary is time independent. Under the dividend parametrization, this is made in many studies, for example, see formula (52) from [Merton \(1973\)](#), formulas (9) and (15) from [Kim \(1990\)](#), Proposition 2.3 from [Jacka \(1991\)](#), Theorem 7.2 from [Karatzas and Shreve \(1998\)](#), or formula (5.1.10) from [Kwok \(2008\)](#). Under the discount parametrization, these results can be found in Theorems 1 and 2 from [Shiryaev et al. \(1995\)](#) and in Theorems 6.1 and 6.2 of [Zaevski \(2021\)](#). If we denote by $V_a(\cdot)$ the price function, we can write for a put-styled option

$$V_a(x) := \left(\frac{K}{q+1}\right)^{q+1} \left(\frac{q}{x}\right)^q. \quad (38)$$

Particularly, for $x = K$, we define the important value $\bar{\eta}$,

$$\bar{\eta} := V_a(K) = K \frac{q^q}{(q+1)^{q+1}}. \quad (39)$$

4.1 The main results

As we mentioned above, the holder's optimal set is an interval $(0, A)$ for some constant A not above the strike, $A \leq K$. The possible form of the writer's one is more complicated – it may be the empty set, the singleton $\{K\}$, or an interval $[B, K]$, $0 < A < B < K$. These results for the put options with fixed penalties and without dividends are obtained in [Kyprianou \(2004\)](#), and under the dividend parametrization in [Ekström \(2006\)](#) and [Suzuki and Sawaki \(2007\)](#). On the other hand, the three-component penalties are considered under the discount parametrization in [Zaevski](#)

(2023). We are interested in which values of the penalty coefficients η_1 (proportion), η_2 (shares of underlying), and η_3 (fixed amount) lead to which case.

Let us define the constants ξ_1 , ξ_2 , and L as

$$\begin{aligned}\xi_1 &:= \eta_1 - \eta_2, \\ \xi_2 &:= \eta_2 + \frac{\eta_3}{K}, \\ L &= \xi_2 K = \eta_2 K + \eta_3.\end{aligned}\tag{40}$$

The constant L plays an outstanding role in this study. Defined in that way, it is the amount that the writer owes if he cancels the option at the strike. This strategy is very important as far as the strike belongs to the writer's optimal set if it is not empty. In this light, the option type can be recognized through two criteria: (A) whether the strike is writer-optimal, and (B) whether all points below the strike are not. Criterion (A) is met when the price of a regular perpetual American option under the assumption $S_0 = K$ is higher than the financial result of the immediate canceling, i.e. L . It turns out that criterion (B) is related to the left derivative of the price function of the L -American option taken in the strike.

Note that $\xi_1 + \xi_2 \geq 1$ since $\eta_1 \geq 1$. It is proven in Proposition 4.2 of Zaeviski (2023) that canceling is never optimal for the writer if $\eta_2 \geq \eta_1$. Thus we assume $\xi_1 > 0$, hereafter. We shall prove a stronger condition.

Proposition 4.1. *The cancellation is never optimal for the writer if and only if $L > \bar{\eta}$.*

Proof. First, suppose that the writer's optimal set is empty, i.e doing nothing is the best writer's strategy. Hence $V(x) = V_a(x)$. Particularly, for $x = K$, we have $n_2(K) > V(K) = V_a(K)$. Formulas (39) and (40) lead to the desired result.

Suppose now that $L > \bar{\eta}$ or equivalently $V_a(K) < n_2(K)$. We shall prove that canceling is never optimal applying an approach similar to the one used in Lemma 3.1 of Suzuki and Sawaki (2007). Let the function $U(x)$ be defined as

$$U(x) = V_a(x) - n_2(x).\tag{41}$$

Having in mind that the optimal boundary of the regular American options is $\frac{q}{q+1}K$, we derive the derivative of function (41):

$$U'(x) = \begin{cases} \eta_1 - \eta_2 - 1, & \text{if } x \leq \frac{q}{q+1}K, \\ \eta_1 - \eta_2 - \left(\frac{q}{q+1} \frac{K}{x}\right)^{q+1}, & \text{if } x \in \left(\frac{q}{q+1}K, K\right), \\ -\eta_2 - \left(\frac{q}{q+1} \frac{K}{x}\right)^{q+1}, & \text{if } x > K. \end{cases}\tag{42}$$

First, note that $U'(x) < 0$ whenever $x > K$. Furthermore, if

$$\eta_1 < \eta_2 + \left(\frac{q}{q+1} \frac{K}{x}\right)^{q+1},\tag{43}$$

then $U'(x)$ is always negative. On the other hand, if the relation opposite to (43) holds, then $U'(x)$ has a unique root less than K . Furthermore, $U'(x)$ is negative before it

and positive after. In all cases, the function $U'(x)$ achieves its maximum either for $x = 0$ or for $x = K$. Having in mind that

$$\begin{aligned} U(0) &= -(\eta_1 - 1)K - \eta_3, \\ U(K) &= V_a(K) - n_2(K), \end{aligned} \quad (44)$$

we conclude that $U(x) < 0$ for every $x > 0$ and thus $V(x) \leq V_a(x) < n_2(x)$. Thus we conclude that the writer's optimal set is empty. $\square$

Remark 4. The inequality $\xi_1 \leq 0$ is stronger than $L > \bar{\eta}$ since $\eta_1 \geq 1 > \frac{\bar{\eta}}{K}$.

Remark 5. Having in mind that the strategy of the first hit to the strike is possible for the writer and the payoff at the strike is namely L , we can conclude that $V(x) \leq V_L(x)$. Furthermore, a cancellable option is of L -American style if and only if $V(x) = V_L(x)$.

We assume $L < \bar{\eta}$, hereafter. We shall provide now a theorem which characterizes the penalty values for which the writer's optimal set is an interval instead of the singleton $\{K\}$. Thus the option turns from L -American into real cancellable.

Theorem 4.2. *The option is real cancellable if and only if*

$$V'_L(K^-) < -\xi_1. \quad (45)$$

Proof. Suppose first that the inequality (45) holds. Having in mind $\frac{d}{dx}n_2(K^-) = -\xi_1$, we conclude that the function

$$f(x) = V_L(x) - n_2(x) \quad (46)$$

is left-decreasing at the point K and $f(K) = 0$. Therefore, there exists an interval (k_1, K) such that

$$V_L(x) > n_2(x) \geq V(x). \quad (47)$$

Combining inequality (47) with Remark 5, we conclude that the option is real cancellable since it cannot be regular American when $L < \bar{\eta}$.

Suppose now that the inequality (45) does not hold. Hence, $f'(K^-) > 0$ and therefore function (46) is left-increasing in the point K . Having in mind that $f(K) = 0$, we conclude that there exists an interval (k_1, K) in which $f(K) < 0$ and thus

$$V_L(x) \leq n_2(x) \quad \forall x \in (k_1, K). \quad (48)$$

Suppose that there exists a writer's optimal point $k_2 \in (k_1, K)$. Therefore, all points $y \in (k_2, K)$ are writer's optimal too. Combining inequality (48) with Remark 5, we conclude

$$V(y) \leq V_L(y) \leq n_2(y) = V(y) \quad (49)$$

for $y \in (k_2, K)$ and therefore $V_L(y) = n_2(y)$ in this interval. However, we can easily check that this is impossible due to formula (31). This finishes the proof. $\square$

4.2 Necessary and sufficient conditions

Suppose that the option is not regular American, i.e. $L < \bar{\eta}$. We shall obtain now some necessary and sufficient conditions for the coefficients η_1 , η_2 , and η_3 that recognize the type of the option – L -American or real cancellable. We shall work under the following scheme:

1. We obtain a condition alternative to (45) for the option to be real cancellable. It says that a suitable function $g(\cdot)$ taken at the point a^* is positive, where a^* is the root of function (30) and it determines the optimal boundary of an L -American option; the function $g(\cdot)$ is defined in (51) (Propositions 4.3 and 4.4).
2. We prove that the option can be real cancellable only when $r < 0$ (Proposition 4.6, see also Remark 7).
3. We investigate the possible behaviors of the function $g(\cdot)$ (Lemma 4.5 and Proposition 4.7).
4. Furthermore, we determine the critical values for ξ_1 and ξ_2 below which the positive domain of the function $g(\cdot)$ (i.e. the set of the inputs that lead to positive function values) is not empty. Note that the critical value for ξ_2 depends on ξ_1 (Corollary 4.8, Propositions 4.9, and 4.10).
5. This step is very important. We prove that if the function $g(\cdot)$ has a positive domain, then the point a^* belongs to it. Thus the condition for the option to be real cancellable turns to checking when the function $g(\cdot)$ has positive values somewhere in the interval $(0, 1]$ (Proposition 4.11).
6. We prove several auxiliary results that give some relations between the triples leading to real cancellable options (Propositions 4.13, 4.14, and Lemma 4.15).
7. We summarize all these results in a theorem that categorizes all possible cases (Theorem 4.16).

Suppose that the holder's optimal boundary of an L -American option is A^* and $A^* \leq S_0 = x \leq K$. The following proposition determines the boundary through the value of the amount payable at the strike.

Proposition 4.3. *Let the constant ξ_2 be defined by (40), the function $h(\cdot; \cdot)$ by (30), and $a^*(\xi_2) := \frac{A^*}{K}$. Note that the dependence of $a^*(\cdot)$ on ξ_2 comes from A^* . We have*

$$h(a^*(\xi_2), \xi_2) = 0. \quad (50)$$

Proof. Based on Proposition 3.5, we conclude that for a fixed ξ_2 , condition (50) is necessary and sufficient for the optimal boundary of the L -American option to be $Ka^*(\xi_2)$. $\square$

Let the function $g(a; \xi_1, \xi_2)$ be defined as

$$g(a; \xi_1, \xi_2) = a^p (\xi_1 - q\xi_2) - a^{q+1}p + a^q p - (\xi_1 + \xi_2(p - q)). \quad (51)$$

Based on Theorem 4.2, we can obtain the following condition for an option to be real cancellable.

Proposition 4.4. *The option is real cancellable if and only if $g(a^*(\xi_2); \xi_1, \xi_2) > 0$.*

Proof. Theorem 4.2 requires checking of inequality (45). The left derivative of the price function (the second statement of formula (31)) in the point K is

$$V'_L(K) = \frac{A^{*p} q \xi_2 + A^{*q+1} p K^{p-q-1} - A^{*q} p K^{p-q} + K^p \xi_2 (p-q)}{K^p - A^{*p}}. \quad (52)$$

We finish the proof by several simple calculations having in mind $A^* = K a^*(\xi_2)$. $\square$

Propositions 4.3 and 4.4 show that we need the behavior of functions $h(a; \cdot, \cdot)$ and $g(a; \cdot, \cdot)$ in the interval $a \in (0, 1]$. More precisely, we need to find the positive domain of function (51) in this interval. The derivative of function $g(a; \cdot, \cdot)$ can be presented as

$$g_a(a; \xi_1, \xi_2) = p a^{q-1} m(a; \xi_1, \xi_2), \quad (53)$$

where $m(\cdot; \cdot, \cdot)$ is

$$m(a; \xi_1, \xi_2) = a^{p-q} (\xi_1 - q \xi_2) - a(q+1) + q. \quad (54)$$

The possible behavior of function $g(a; \xi_1, \xi_2)$ is provided in the following lemma.

Lemma 4.5. *The endpoints of the function $g(a; \xi_1, \xi_2)$ are negative:*

$$\begin{aligned} g(0; \xi_1, \xi_2) &= -(\xi_1 + \xi_2(p-q)), \\ g(1; \xi_1, \xi_2) &= -p \xi_2. \end{aligned} \quad (55)$$

The function exhibits one of the following three behaviors in the interval $a \in (0, 1]$:

- (A) *Increasing negative function.*
- (B) *Inverted U-shaped function – first increases and then decreases.*
- (C) *In addition to the second case, the function has a local negative minimum after the maximum.*

Proof. The proof is provided in Appendix A. $\square$

The following proposition shows that the option can be real cancellable only when $r < 0$.

Proposition 4.6. *If $r \geq 0$, then $g(a; \xi_1, \xi_2) < 0$ for every $a \in (0, 1]$.*

Proof. The proof is provided in Appendix A. $\square$

Assume now that $r < 0$ or equivalently $p > 2q + 1$. Having in mind Lemma 4.5, we conclude that the option can be real cancellable only when one of the cases (B) and (C) holds – note that this condition is only necessary. Below we discuss when this happens.

Proposition 4.7. *One of the cases (B) and (C) holds if and only if the inequality*

$$\xi_1 - q \xi_2 < l \quad (56)$$

holds, where the constant l is defined by formula (22).

Proof. The proof is provided in Appendix A. $\square$

We need to strengthen condition (56) in a way that would allow later to derive the critical value for ξ_1 as a root of a decreasing function in a certain interval.

Corollary 4.8. *If the positive domain of the function $g(a; \xi_1, \xi_2)$ is not empty, then $\xi_1 < l$, where l is defined by formula (22).*

Proof. Suppose that $\xi_1 \geq l$. Note that $\xi_1 > 1$ due to Lemma 2.4. Therefore, the triple $\{\xi_1, 0, 0\}$ leads to the case (A) due to Proposition 4.7 – note that condition (56) turns namely into $\xi_1 < l$. Thus the function $g(a; \xi_1, 0)$ is negative. But $g(a; \xi_1, \xi_2)$ decreases w.r.t. ξ_2 and hence it is always negative. Therefore, its positive domain is empty. $\square$

We continue our investigation on the positive domain of the function $g(a; \xi_1, \xi_2)$ that gives possibilities for a real cancellable feature of the option. Remind that one of the cases (B) or (C) holds. We need some additional notations. Let the function $g(a; \xi_1, \xi_2)$ achieves its maximum at the point $\bar{a}_1(\xi_1, \xi_2)$ and its possible minimum at $\bar{a}_2(\xi_1, \xi_2)$. If this minimum does not exist, then we set $\bar{a}_2(\xi_1, \xi_2) = 1$. If $g(\bar{a}_1(\xi_1, \xi_2); \xi_1, \xi_2) > 0$ then the function $g(a; \xi_1, \xi_2)$ has two roots – we denote them by $a_1(\xi_1, \xi_2)$ and $a_2(\xi_1, \xi_2)$. We shall prove now that the positive domain of the function $g(a; \xi_1, \xi_2)$ is not empty for small enough values of ξ_1 and ξ_2 , i.e. $g(\bar{a}_1(\xi_1, \xi_2); \xi_1, \xi_2) > 0$. Furthermore, we shall show that $g(a^*(\xi_2); \xi_1, \xi_2) > 0$ for these values of ξ_1 and ξ_2 . This statement is of outstanding importance. It says that if the positive domain of $g(\cdot)$ is nonempty, then the optimal point $a^*(\xi_2)$ is always in it, i.e. the option is real cancellable only when $g(\bar{a}_1(\xi_1, \xi_2); \xi_1, \xi_2) > 0$. As a consequence, $a^*(\xi_2) = \bar{a}_1(\xi_1, \xi_2)$ when $g(\bar{a}_1(\xi_1, \xi_2); \xi_1, \xi_2) = 0$.

Proposition 4.9. *The function $g(\bar{a}_1(\xi_1, 0); \xi_1, 0)$ is decreasing w.r.t. $\xi_1 \in (0, l)$ and changes its sign in the interval $(1, l)$, where $l > 1$ is defined by formula (22). Thus if ξ_1^* is the solution of $g(\bar{a}_1(\xi_1, 0); \xi_1, 0) = 0$, then $\xi_1^* \in (1, l)$ and $g(\bar{a}_1(\xi_1, 0); \xi_1, 0) > 0$ for every $\xi_1 < \xi_1^*$.*

Proof. The proof is provided in Appendix A. $\square$

Proposition 4.10. *If $\xi_1 < \xi_1^*$, then the function $g(\bar{a}_1(\xi_1, \xi_2); \xi_1, \xi_2)$ decreases w.r.t. ξ_2 and changes its sign in the interval $(0, \bar{\xi}_2)$, where $\bar{\xi}_2$ is*

$$\bar{\xi}_2 := \frac{\bar{\eta}}{K} = \frac{q^q}{(q+1)^{q+1}}. \quad (57)$$

Thus, if $\xi_2^(\xi_1)$ is the solution of $g(\bar{a}_1(\xi_1, \xi_2); \xi_1, \xi_2) = 0$, then $\xi_2^*(\xi_1) \in (0, \bar{\xi}_2)$ and $g(\bar{a}_1(\xi_1, \xi_2); \xi_1, \xi_2) > 0$ for every $\xi_2 < \xi_2^*(\xi_1)$.*

Proof. The proof is provided in Appendix A. $\square$

Proposition 4.11. *If ξ_1 and ξ_2 are such that $g(\bar{a}_1(\xi_2; \xi_1); \xi_1, \xi_2) > 0$, then $g(a^*(\xi_2); \xi_1, \xi_2) > 0$.*

Proof. The proof is provided in Appendix A. $\square$

Next we shall establish a result which gives that if an option is real cancellable then all options with lower in some sense penalties are real cancellable too. To do this, we use the traditional definition for vector ordering.

Definition 4.12. A triple of reals is less than another if all its elements are not higher than the corresponding ones of the second triple and at least one is lower.

Proposition 4.13. *If a triple $\{\eta_1, \eta_2, \eta_3\}$ leads to a real cancellable option, then all triples less than it lead again to real cancellable options.*

Proof. We can rewrite function (51) as

$$g(a; \xi_1, \xi_2) = -\eta_1(1 - a^p) - \eta_2[a^p(q+1) + p - q - 1] - \frac{\eta_3}{K}(-a^p q + p - q) - a^{q+1}p + a^q p. \quad (58)$$

Therefore, function (58) decreases w.r.t η_1 , η_2 , and η_3 . Propositions 4.4 and 4.11 show that the positive domain of function (58) is not empty and thus the positive domains for all triples less than $\{\eta_1, \eta_2, \eta_3\}$ are not empty too. The same propositions prove the desired result. $\square$

We continue by characterizing the set of triples $\{\eta_1, \eta_2, \eta_3\}$ that lead to real cancellable options.

Proposition 4.14. *If a triple $\{\eta_1, \eta_2, \eta_3\}$ leads to a real cancellable option, then $\eta_1 < \xi_1^*$, where ξ_1^* is defined in Proposition 4.9.*

Proof. Suppose that the triple $\{\eta_1, \eta_2, \eta_3\}$ leads to a real cancellable option. Proposition 4.13 shows that the triple $\{\eta_1, 0, 0\}$ leads to a real cancellable option too. Proposition 4.9 shows that $\eta_1 < \xi_1^*$. $\square$

Before to establish the main result for the put options, we need the following lemma.

Lemma 4.15. *Let $\eta_1 \in [1, \xi_1^*)$ and the function $f(\cdot)$ be defined in the interval $(0, \bar{\xi}_2)$ as*

$$f(x) = \xi_2^*(\eta_1 - x) - x. \quad (59)$$

Note that $\xi_2^(\eta_1 - x)$ means the function $\xi_2^*(\cdot)$ taken in the point $\eta_1 - x$. Under these assumptions, $f(x)$ is a decreasing function, $f(0) > 0$, and $f(\bar{\xi}_2) < 0$. Thus the equation $f(x) = 0$ has a unique root in the interval $(0, \bar{\xi}_2)$ below which $f(\cdot)$ is positive. We shall denote this root by $\eta_2^*(\eta_1)$.*

Proof. The proof is provided in Appendix A. $\square$

We can summarize the derived results: the large enough penalties lead to regular American options (Proposition 4.1); the low enough penalties lead to real cancellable options equivalently to a nonempty positive domain of the function $g(a; \xi_1, \xi_2)$; and the middle values lead to L -American options. The precise results are given in the following theorem.

Theorem 4.16. *Let a cancellable American put option has the penalty structure $\{\eta_1, \eta_2, \eta_3\}$. The following statements characterize it:*

1. If $r \geq 0$ then the option is L -American for $L < \bar{\eta}$ and regular American otherwise.
2. Suppose that $r < 0$. Let the pair (α_1, ξ_1^*) be the solution of the system

$$\begin{aligned} g(a; \xi_1, 0) &= 0, \\ m(a; \xi_1, 0) &= 0, \end{aligned} \tag{60}$$

where the functions $g(\cdot)$ and $m(\cdot)$ are defined by formulas (51) and (54). The solution exists, it is unique, $\alpha_1 < 1$, and $1 < \xi_1^* < l$. If $\bar{\xi}_1 < \xi_1^*$, then the system

$$\begin{aligned} g(a; \bar{\xi}_1, \xi_2) &= 0, \\ m(a; \bar{\xi}_1, \xi_2) &= 0 \end{aligned} \tag{61}$$

has at most two solutions – we denote by $(\alpha_2(\bar{\xi}_1), \xi_2^*(\bar{\xi}_1))$ the lower one w.r.t. the variable a . We have $\alpha_2(\bar{\xi}_1) < 1$ and $0 < \xi_2^*(\bar{\xi}_1) < \frac{\bar{\eta}}{K}$. Hence:

(a) The option is real cancellable when

$$\{\eta_1 < \xi_1^*, \eta_2 < \eta_2^*(\eta_1), \eta_3 < K[\xi_2^*(\eta_1 - \eta_2) - \eta_2]\}. \tag{62}$$

Note that $L < \bar{\eta}$ since $\xi_2^*(\eta_1 - \eta_2) < \bar{\xi}_2$, see Proposition 4.10.

(b) It is L -American if $L < \bar{\eta}$ and at least one of the requirements (62) does not hold.

(c) It is regular American for $L \geq \bar{\eta}$.

Note that $\xi_2^*(\eta_1 - \eta_2) > \eta_2$ when $\eta_2 < \eta_2^*(\eta_1)$ due to Lemma 4.15.

Proof. System (60) means that the function $g(\cdot; \xi_1^*, 0)$ has an extremum at the point α_1 and its value is zero. Note that this extremum is the maximum. If we suppose that it is the minimum in the case (C) of Lemma 4.5, then $g(1; \xi_1^*, 0)$ has to be positive, which is impossible. Having in mind Propositions 4.9 and 4.11, we conclude that the positive domain of the function $g(\cdot; \xi_1, 0)$ is not empty if and only if $\xi_1 < \xi_1^*$. Furthermore, $g(a^*(0); \xi_1, 0) > 0$. Also, $1 < \xi_1^* < l$ due to Proposition 4.9.

Analogously, system (61) shows that the function $g(\cdot; \bar{\xi}_1, \xi_2^*(\bar{\xi}_1))$ has a maximum at the point α_2 and its value is zero. Propositions 4.10 and 4.11 show that the positive domain of the function $g(\cdot; \bar{\xi}_1, \xi_2)$ is not empty if and only if $\xi_2 < \xi_2^*(\bar{\xi}_1)$. Furthermore, $g(a^*(\xi_2); \bar{\xi}_1, \xi_2) > 0$. Note that $0 < \xi_2^*(\bar{\xi}_1) < \frac{\bar{\eta}}{K}$ due to Proposition 4.10. $\square$

Some calculations lead to the following method for deriving the critical values.

Corollary 4.17. *Let the functions $F(a, \xi_1)$, $H(a)$, and $G(a, \xi_1)$ be defined as*

$$\begin{aligned} F(a, \xi_1) &= \frac{a^{p-q} \xi_1 - a(q+1) + q}{a^{p-q}}, \\ H(a) &= -a^{p+1}(p-q-1) + a^p(p-q) - a(q+1) + q, \\ G(a, \xi_1) &= -a^{p+1}q(p-q-1) + a^p q(p-q) - a^{p-q} p \xi_1 \\ &\quad + a(p-q)(q+1) - q(p-q). \end{aligned} \quad (63)$$

We have

$$\begin{aligned} \xi_1^* &= -F(\alpha_1, 0), \\ \xi_2^*(\bar{\xi}_1) &= \frac{F(\alpha_2, \bar{\xi}_1)}{q}, \end{aligned} \quad (64)$$

where α_1 and α_2 are the solutions of the equations $H(a) = 0$ and $G(a, \bar{\xi}_1) = 0$, respectively. In addition, we have to impose the condition

$$m_a(\alpha_2, \bar{\xi}_1, \xi_2^*(\bar{\xi}_1)) < 0 \quad (65)$$

to avoid the possible minimum of the function $g(a, \bar{\xi}_1, \xi_2^*(\bar{\xi}_1))$ if the case (C) holds. Note that this is not necessary for α_1 since the case (B) holds when $\xi_2 = 0$.

Proof. System (60) can be rewritten as

$$\begin{aligned} \xi_1 &= \frac{a(q+1) - q}{a^{p-q}}, \\ \xi_1 &= pa^q \frac{1-a}{1-a^p}, \end{aligned} \quad (66)$$

which proves the first result. The second one holds due to the following presentation of system (61):

$$\begin{aligned} \xi_2 &= \frac{a^{p-q} \bar{\xi}_1 - a(q+1) + q}{qa^{p-q}}, \\ \xi_2 &= \frac{a^p \bar{\xi}_1 - a^{q+1} p + a^q p - \bar{\xi}_1}{p-q+qa^p}. \end{aligned} \quad (67)$$

□

Remark 6. Let us discuss the mechanism for recognizing the option's type. The option is regular American if $L \geq \bar{\eta}$. If the opposite relation holds, then we derive the critical value for ξ_1 , namely ξ_1^* . Based on it, for every $\bar{\xi}_1 < \xi_1^*$, we derive the critical value for ξ_2 , namely $\xi_2^*(\bar{\xi}_1)$. If $\xi_1 < \xi_1^*$ and $\xi_2 < \xi_2^*(\bar{\xi}_1)$, then the option is real cancellable. If one of these inequalities does not hold, then we have an L -American option.

Based on Theorem 4.16 and Remark 6, we summarize how the option changes its type when the penalty parameters are passing through their critical levels.

1. If the second and third components are large enough, i.e. $\eta_2 K + \eta_3 \geq \bar{\eta}$ (equivalent to $L \geq \bar{\eta}$), then the option is regular American. Note that the component η_1 does not influence this type.
2. Suppose that $\eta_2 K + \eta_3 < \bar{\eta}$. Now η_1 has its impact. We calculate its critical value ξ_1^* .
 - (a) If $\eta_1 \geq \xi_1^*$, then we have an L -American option.
 - (b) If $1 \leq \eta_1 < \xi_1^*$, then we obtain the critical value for η_2 that depends on η_1 , i.e. $\eta_2^*(\eta_1)$.
 - i. If η_2 is such that $\eta_2^*(\eta_1) \leq \eta_2 < \frac{\bar{\eta}}{K}$, then we have an L -American option. Note that $\eta_2^*(\eta_1) \leq \frac{\bar{\eta}}{K}$ when $1 \leq \eta_1 < \xi_1^*$ due to Lemma 4.15.
 - ii. If $\eta_2 < \eta_2^*(\eta_1)$, then we obtain the critical value for η_3 , it is $\eta_3^*(\eta_1, \eta_2) = K [\xi_2^*(\eta_1 - \eta_2) - \eta_2]$. We have $\eta_3^*(\eta_1, \eta_2) > 0$ when $\eta_2 < \eta_2^*(\eta_1)$ due to Lemma 4.15.
 - A. If $\eta_3^*(\eta_1, \eta_2) \leq \eta_3 < \bar{\eta} - \eta_2 K$, then we have an L -American option. Note that Proposition 4.10 shows that $\eta_3^*(\eta_1, \eta_2) < \bar{\eta} - \eta_2 K$.
 - B. If $\eta_3 < \eta_3^*(\eta_1, \eta_2)$, then we have a real cancellable option.

Remark 7. Let us discuss briefly why an option cannot be real cancellable when $r \geq 0$. Suppose the opposite, i.e. there exists a value $k_1 < K$ such that it is writer's optimal. Hence, the interval $[k_1, K]$ belongs to the writer's optimal set Y^s . Similar arguments that stand behind Lemma 3.2 show that $(\mathcal{B}n_2)(x) > 0$ in the interval (k_1, K) , where the function $n_2(\cdot)$ is given in (4) and the operator $\mathcal{B}$ is defined by formula (27). We can motivate this by the following intuitive construction. Let for an arbitrary time value s , $\tau(s)$ be the lower of the first exit of the underlying asset from the strip (k_1, K) and s . For an arbitrary starting point $x \in (k_1, K)$, this strategy would give a worse financial result for the writer than the immediate canceling. Having in mind that the exercise is not optimal for the holder in the strip (k_1, K) and applying the Dynkin formula, we conclude for the result of the strategy $\tau(s)$:

$$\mathbb{E}^x \left[e^{-(r+\lambda)\tau(s)} n_2(S_{\tau(s)}) \right] = n_2(x) + \mathbb{E}^x \left[\int_0^{\tau(s)} (\mathcal{B}n_2)(S_u) du \right] > n_2(x). \quad (68)$$

Taking the limit as $s \rightarrow 0$, we convinced that indeed $(\mathcal{B}n_2)(x) > 0$. In financial terms, this means that if the immediate cancelling is preferable for the writer than keeping the option alive for an infinitesimal period, then $(\mathcal{B}n_2)(x) > 0$. However, this inequality is possible below the strike only when $r < 0$ since

$$\begin{aligned} (\mathcal{B}n_2)(x) &= \lambda(\eta_1 - \eta_2)x - (r + \lambda)(\eta_1 K + \eta_3), \\ (\mathcal{B}n_2)(K) &= -K(r\eta_1 + \lambda\eta_2) - (r + \lambda)\eta_3. \end{aligned} \quad (69)$$

Note that this construction is impossible if the writer's optimal set is the singleton $\{K\}$, because the differentiability of the function $n_2(\cdot)$ is broken in the strike.

5 Call options

We consider now the cancellable call options through some symmetrical arguments. Some proofs will be omitted since they are similar to the put versions. The shape of the optimal sets for the calls is symmetric w.r.t. the strike to those for the puts. The respective results for the fixed penalties under the dividend parametrization are obtained in Kunita and Seko (2004), Ekström and Villeneuve (2006), Emmerling (2012), and Yam et al. (2014) whereas proportional to the usual payoff penalties are considered in Ekström and Villeneuve (2006). The options with three-component penalties are examined in Zaevski (2023). Note that the case $\lambda = 0$ is special – the early exercise is never optimal for the option's holder. All necessary results in this case are obtained in Theorem 3.9 from the same work.

Suppose now that $\lambda > 0$ or equivalently $p > q + 1$. Note that we have to consider the related functions in the interval $(1, \infty)$ instead of $(0, 1)$ since the possible exercise boundaries are above the strike. The price of the perpetual American call option when $S_0 = K$ is $\bar{\eta} := K\bar{\xi}_2$, where

$$\bar{\xi}_2 := \frac{1}{p - q} \left(\frac{p - q - 1}{p - q} \right)^{p - q}. \quad (70)$$

The holder's optimal set is an interval (A, ∞) for some constant A not below the strike. The writer's one can be the empty set, the singleton $\{K\}$, or an interval $[K, B]$, $K < B < A$. The constants ξ_1 and ξ_2 are defined now as

$$\begin{aligned} \xi_1 &:= \eta_1 + \eta_2, \\ \xi_2 &:= \eta_2 + \frac{\eta_3}{K}. \end{aligned} \quad (71)$$

Note that the constant L keeps its value. It is proven in Proposition 3.2 of Zaevski (2023) that canceling is never optimal for the writer if $\eta_3 \geq \eta_1 K$. Thus we consider only the values of ξ_1 and ξ_2 such that $\xi_1 > \xi_2$. The restriction presented in Proposition 4.1 also holds but with the actual value of $\bar{\eta}$. Furthermore, the analogue of Theorem 4.2 gives the criteria for the option to be real cancellable. We summarize these results in the following theorem.

Theorem 5.1. *If $L \geq \bar{\eta}$, then the option is regular American. On the contrary, if $L < \bar{\eta}$, then it is real cancellable if*

$$V'_L(K^+) > \xi_1, \quad (72)$$

and L -American otherwise.

Proof. See the proofs of Proposition 4.1 and Theorem 4.2. □

Having in mind that the function $h(\cdot)$ is taken for $\xi = -\frac{L}{K}$, we see that Proposition 4.3 still holds. Theorem 5.1 shows that we have to find the right derivative of the price function (36) at the strike:

$$V'_L(K^+) = \frac{-a^{*p} q \xi_2 + a^{*q+1} p - a^{*q} p - \xi_2 (p - q)}{a^{*p} - 1}. \quad (73)$$

We shall proceed further using the method presented in the beginning of Section 4.2. Proposition 4.4 is true for the function

$$g(a; \xi_1, \xi_2) = -a^p (\xi_1 + q\xi_2) + a^{q+1}p - a^q p + \xi_1 - \xi_2 (p - q). \quad (74)$$

We need to know when the function $g(\cdot)$ has a positive domain larger than one, i.e. when inputs larger than one make the function positive. The function related to its derivative $m(a; \xi_1, \xi_2)$ now takes the form

$$m(a; \xi_1, \xi_2) = -a^{p-q} (\xi_1 + q\xi_2) + a(q+1) - q. \quad (75)$$

Its endpoints are always negative except in the limiting case since $\xi \geq 1$ and $\xi_2 \geq 0$. The behavior of function (74) is similar to the put case considered in Lemma 4.5 – we can recognize the following three cases:

- (A) Decreasing negative function.
- (B) Inverted U -shaped function – first increase and then decrease.
- (C) In addition to the second case, the function has a local negative minimum before the maximum.

The sign of the risk free rate is again important. The analogue of Proposition 4.6 is as follows.

Proposition 5.2. *If $r \leq 0$, then $g(a; \xi_1, \xi_2) < 0$ for every $a > 1$. Thus the option is L -American when $L < \bar{\eta}$ and regular American when $L \geq \bar{\eta}$.*

Proof. The proof is provided in Appendix A. □

In addition to this proposition, we can provide financial arguments similar to those in Remark 7 why the option cannot be real cancellable when $r < 0$. The important term $(\mathcal{B}n_2)(K)$ now is $(\mathcal{B}n_2)(K) = K(r\eta_1 - \lambda\eta_2) - (r + \lambda)\eta_3$ and it can be positive only when $r > 0$.

Suppose now that $r > 0$ or equivalently $p < 2q + 1$. The condition obtained in Proposition 4.7 can be rewritten as follows.

Proposition 5.3. *The necessary and sufficient condition for one of the cases (B) or (C) to hold is the inequality $\xi_1 + q\xi_2 < l$, where the constant l is defined by formula (22).*

Proof. The proof is provided in Appendix A. □

The next step is to prove that if the function $g(\cdot)$ has a positive domain, then the point a^* belongs to it. Let us keep the meaning of $\bar{a}_1(\xi_1, \xi_2) > \bar{a}_2(\xi_1, \xi_2)$, i.e. the function $g(a; \xi_1, \xi_2)$ achieves its maximum and minimum at these points, respectively. If $g(\bar{a}_1(\xi_1, \xi_2); \xi_1, \xi_2) > 0$, then the function $g(a; \xi_1, \xi_2)$ has two roots: $a_1(\xi_1, \xi_2)$ and $a_2(\xi_1, \xi_2)$. The analogues of Propositions 4.9 and 4.10 and their proofs are identical to the original ones and we omit them. The important Proposition 4.11 still holds – the unique difference in the proof is in the presentation (98) of the function $h(a; \xi_2)$. In the call case, it is

$$h(a; \xi_2) = -a^{p-q}g(a; \xi_1, \xi_2) + \frac{a^p - 1}{pa^{q-1}}g_a(a; \xi_1, \xi_2). \quad (76)$$

Thus we reach the corresponding results for the call options.

Theorem 5.4. 1. If $r \leq 0$, then the game option is L -American for $L < \bar{\eta}$ and regular American otherwise.

2. Suppose that $r > 0$. The solution of system (60), (α_1, ξ_1^*) , exists, $\alpha_1 > 1$, and $1 < \xi_1^* < l$. The functions $g(\cdot)$ and $m(\cdot)$ are defined by formulas (74) and (75). Let $\bar{\xi}_1 \in [1, \xi_1^*]$. The solution of system (61), $(\alpha_2(\bar{\xi}_1), \xi_2^*(\bar{\xi}_1))$, exists, $\alpha_2 > 1$, and $0 < \xi_2^*(\bar{\xi}_1) < \frac{\bar{\eta}}{K}$. Let $\eta_1 \in [1, \xi_1^*]$ and the analogue of function (59) be defined as

$$f(x) := \xi_2^*(\eta_1 + x) - x. \quad (77)$$

We shall denote by $\eta_2^*(\eta_1)$ its solution in the interval $(0, \min\{\xi_1^* - \eta_1, \bar{\xi}_2\})$.² Note that $\eta_1 + \eta_2^*(\eta_1) < \xi_1^*$. The following statements describe the option's essence:

(a) The option is real cancellable when

$$\{\eta_1 < \xi_1^*, \eta_2 < \eta_2^*(\eta_1), \eta_3 < K[\xi_2^*(\eta_1 + \eta_2) - \eta_2]\}. \quad (78)$$

(b) It is L -American if $L < \bar{\eta}$ and at least one of the requirements (78) does not hold.

(c) It is regular American for $L \geq \bar{\eta}$.

Let the functions $F(a, \xi_1)$, $H(a)$, and $G(a, \xi_1)$ be defined by formulas (63), α_1 and α_2 be the roots of the equations $H(a) = 0$ and $G(a, \bar{\xi}_1) = 0$ in the interval $a \in (1, \infty)$. Note that they exist. The critical values can be derived as $\xi_1^* = -F(\alpha_1, 0)$ and $\xi_2^*(\bar{\xi}_1) = -\frac{F(\alpha_2, \bar{\xi}_1)}{q}$. We impose in addition condition (65).

6 Some examples

We present now some examples. Let us consider first put style options with parameters $r = -0.02$, $\lambda = 0.03$, $\sigma = 0.3$, $K = 1$. We chose these values because $r + \lambda > 0$ and $r < 0$, see point one from Theorem 4.16. The value one for the strike is chosen this way to ignore its impact since it can be viewed as a scaling parameter. The results are visualized in Figure 1a. Theorem 4.16 shows that the triples $\{\eta_1, \eta_2, \eta_3\}$ that lead to real cancellable options are in the pyramid formed by the green points and the point $(1, 0, 0)$. The green points are obtained as follows:

1. Point $(\xi_1^*, 0, 0)$: the value of ξ_1^* is obtained via Proposition 4.9 and it is $\xi_1^* = 1.1744$ for the current parameters.

²Note that function (77) decreases due to presentation (102) of the derivative $(\xi_2^*(x))'$. The inequality $f(0) > 0$ is obvious, whereas $f(\xi_1^* - \eta_1) < 0$ because $\xi_2^*(\xi_1^*) = 0$ and $\eta_1 < \xi_1^*$. The inequality $f(\bar{\xi}_2) < 0$ holds due to the call-analogue of Proposition 4.10.

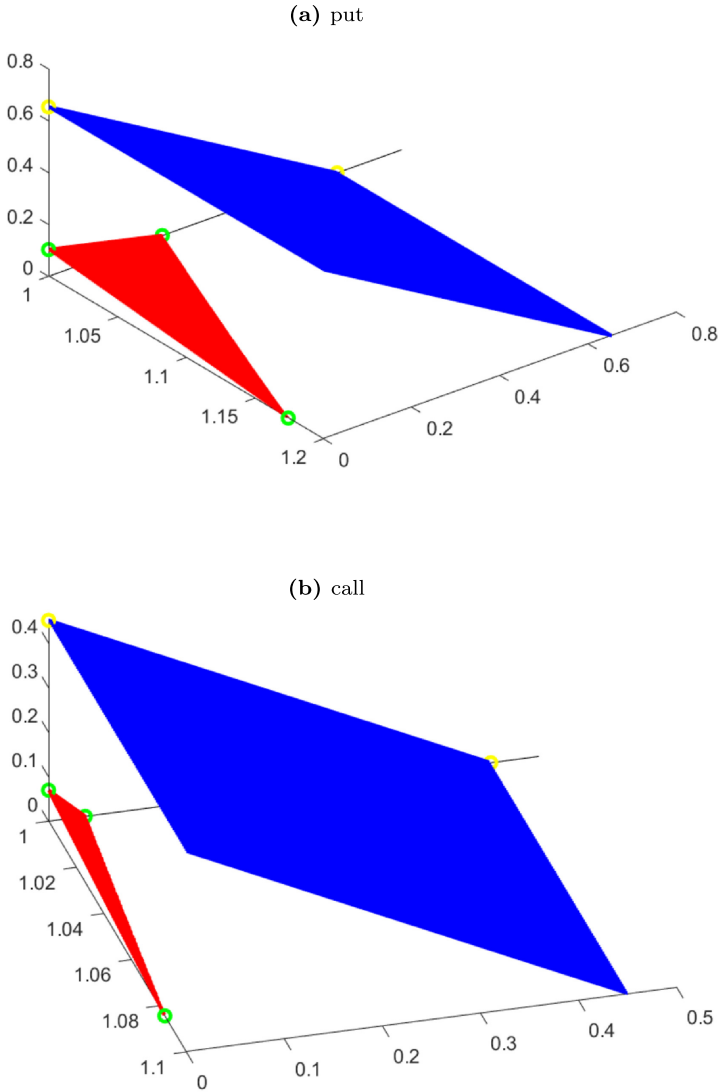


Fig. 1. Restrictions

2. Point $(1, \eta_2^*(1), 0)$: the value of $\eta_2^*(1)$ is obtained via Lemma 4.15 and it is $\eta_2^*(1) = 0.2575$.
3. Point $(1, 0, K\xi_2^*(1))$: the value of $\xi_2^*(1)$ is obtained via Proposition 4.10 and it is $\xi_2^*(1) = 0.1030$.

The value for $L = \eta_2 K + \eta_3$ that distinguishes the L -American options from the regular ones is given by formula (39). Its value is 0.6537, see the yellow points. Thus the triples $\{\eta_1, \eta_2, \eta_3\}$ that lead to an L -American option are in the prism between the plains $\{\eta_1 = 1\}$, $\{\eta_2 = 0\}$, $\{\eta_3 = 0\}$, and the blue one, cut by the above-mentioned

Table 1. Put options

$r = -0.02, \lambda = 0.03, \xi_1^* = 1.1744, \overline{\eta} = 0.6537$					
	$\eta_2^*(\eta_1)$	$\eta_2 = \eta_2^*(\eta_1)$	$\eta_2 = \frac{2}{3}\eta_2^*(\eta_1)$	$\eta_2 = \frac{1}{3}\eta_2^*(\eta_1)$	$\eta_2 = 0$
$\eta_1 = 1$	0.2575	0	0.0341	0.0685	0.1030
$\eta_3 = 1.05$	0.1819	0	0.0243	0.0487	0.0733
$\eta_3 = 1.1$	0.1075	0	0.0145	0.0291	0.0437
$\eta_3 = 1.1744$	0	0	0	0	0
$r = -0.01, \lambda = 0.03, \xi_1^* = 1.0336, \overline{\eta} = 0.5004$					
$\eta_1 = 1$	0.0281	0	0.0049	0.0098	0.0148
$\eta_3 = 1.01$	0.0189	0	0.0034	0.0068	0.0102
$\eta_3 = 1.02$	0.0104	0	0.0019	0.0038	0.0058
$\eta_3 = 1.0336$	0	0	0	0	0
$r = -0.01, \lambda = 0.23, \xi_1^* = 1.0566, \overline{\eta} = 0.6270$					
$\eta_1 = 1$	0.0851	0	0.0107	0.0217	0.0329
$\eta_3 = 1.02$	0.0527	0	0.0069	0.0139	0.0211
$\eta_3 = 1.04$	0.0226	0	0.0031	0.0063	0.0094
$\eta_3 = 1.0566$	0	0	0	0	0

pyramid for the real cancellable options (the red plain). The triples that lead to the regular American options are above the blue plain – they are

$$\eta_2 K + \eta_3 \geq K \frac{q^q}{(q+1)^{q+1}} = 0.6537. \quad (79)$$

Let us consider the call style options. We use the same parameters except the risk-free rate – we assume now that $r = 0.02$ due to the first point of Theorem 5.4. The results are presented in Figure 1b. The critical values that form the pyramid for the real cancellable options are $\xi_1^* = 1.0843$, $\eta_2^*(1) = 0.0374$, and $\xi_2^*(1) = 0.0698$. Critical value (70) for $L = \eta_2 K + \eta_3$ above which the option is real cancellable is 0.4510.

Some particular values are presented for put and call options in Tables 1 and 2, respectively. We give the critical value ξ_1^* for the coefficient η_1 in the head of the tables. Above this level, the option turns from real cancellable into L -American. The critical values for η_2 given η_1 are presented in a separate column. The rest of the tables contain the critical values for η_3 given η_1 and η_2 . The value for L above which the option is regular American, $\overline{\eta}$, is given again in the head of the tables.

A Some proofs

Proof of Lemma 4.5. The derivative of function $m(a; \xi_1, \xi_2)$ is

$$m_a(a; \xi_1, \xi_2) = a^{p-q-1} (p-q) (\xi_1 - q\xi_2) - q - 1. \quad (80)$$

Hence, it can be always negative in the interval $a \in (0, 1]$ or first negative and then positive – note that it is monotone nonetheless increasing or decreasing.

Let us consider the first case. We have that the function $m(a; \xi_1, \xi_2)$ is decreasing with a positive right endpoint and thus it is always positive. Hence, the case (A) holds.

Table 2. Call options

$r = 0.02, \lambda = 0.03, \xi_1^* = 1.0843, \overline{\eta} = 0.4510$					
	$\eta_2^* (\eta_1)$	$\eta_2 = \eta_2^* (\eta_1)$	$\eta_2 = \frac{2}{3} \eta_2^* (\eta_1)$	$\eta_2 = \frac{1}{3} \eta_2^* (\eta_1)$	$\eta_2 = 0$
$\eta_1 = 1$	0.0374	0	0.0231	0.0463	0.0698
$\eta_3 = 1.03$	0.0238	0	0.0145	0.0290	0.0437
$\eta_3 = 1.06$	0.0105	0	0.0063	0.0126	0.0189
$\eta_3 = 1.0843$	0	0	0	0	0
$r = 0.01, \lambda = 0.03, \xi_1^* = 1.0239, \overline{\eta} = 0.4281$					
$\eta_1 = 1$	0.0084	0	0.0045	0.0091	0.0137
$\eta_3 = 1.01$	0.0048	0	0.0025	0.0050	0.0075
$\eta_3 = 1.02$	0.0013	0	0.0006	0.0013	0.0019
$\eta_3 = 1.0239$	0	0	0	0	0
$r = 0.01, \lambda = 0.2, \xi_1^* = 1.0336, \overline{\eta} = 0.5004$					
$\eta_1 = 1$	0.0148	0	0.0092	0.0186	0.0281
$\eta_3 = 1.01$	0.0102	0	0.0062	0.0125	0.0189
$\eta_3 = 1.02$	0.0058	0	0.0034	0.0069	0.0104
$\eta_3 = 1.0336$	0	0	0	0	0

The alternative behavior leads to a U -shape for $m(a; \xi_1, \xi_2)$. Having in mind that $m(0; \xi_1, \xi_2) = q > 0$, we conclude that all of the cases (A), (B), and (C) are possible and which is the actual one is determined by the position of this U -shaped curve w.r.t. the abscissa. $\square$

Proof of Proposition 4.6. Lemma 2.3 shows that the inequality $r \geq 0$ is equivalent to $p \leq 2q + 1$. We shall prove that the function $g(a; \xi_1, \xi_2)$ is negative at its extrema. Suppose that $\alpha \in (0, 1]$ is such an extremum and therefore $m(\alpha; \xi_1, \xi_2) = 0$. Hence, function (51) can be presented in the point α as

$$g(\alpha; \xi_1, \xi_2) = g_1(\alpha; \xi_1, \xi_2) \quad (81)$$

for

$$g_1(a; \xi_1, \xi_2) = a^q [-a(p - q - 1) + p - q] - (\xi_1 + \xi_2(p - q)). \quad (82)$$

Its derivative

$$g'_1(a; \xi_1, \xi_2) = a^{q-1} [-a(q + 1)(p - q - 1) + q(p - q)] \quad (83)$$

is positive since $g'_1(1; \xi_1, \xi_2) = 2q + 1 - p > 0$. Hence,

$$\begin{aligned} g(\alpha; \xi_1, \xi_2) &= g_1(\alpha; \xi_1, \xi_2) \\ &\leq g_1(1; \xi_1, \xi_2) \\ &= 1 - (\xi_1 + \xi_2(p - q)) \leq 1 - (\xi_1 + \xi_2) \leq 0. \end{aligned} \quad (84)$$

Having in mind the inequalities $g(0; \xi_1, \xi_2) < 0$ and $g(1; \xi_1, \xi_2) < 0$, we conclude that the function $g(a; \xi_1, \xi_2)$ is always negative. $\square$

Proof of Proposition 4.7. Suppose that $m_a(1; \xi_1, \xi_2) > 0$ and therefore the zero of the derivative $m_a(a; \xi_1, \xi_2)$, say $\tilde{a}$, is less than one and it is

$$\tilde{a} = \left(\frac{q+1}{(p-q)(\xi_1 - q\xi_2)} \right)^{\frac{1}{p-q-1}}. \quad (85)$$

Hence, the function $m(a; \xi_1, \xi_2)$ is U -shaped. Therefore, the triple $\{\eta_1, \eta_2, \eta_3\}$ leads to the case (B) or (C) if and only if $m(\tilde{a}; \xi_1, \xi_2) < 0$. We can easily check that this is equivalent to inequality (56).

Suppose now that $m_a(1; \xi_1, \xi_2) \leq 0$. Having in mind formula (80), we see that $m_a(a; \xi_1, \xi_2) < 0$ for all $a \in (0, 1]$. In addition, the inequality $m_a(1; \xi_1, \xi_2) < 0$ is equivalent to

$$\xi_1 - q\xi_2 < \frac{q+1}{p-q}. \quad (86)$$

We can easily check that this leads to $m(1; \xi_1, \xi_2) < 0$. This inequality, together with $m(0; \xi_1, \xi_2) > 0$, shows that the case (B) holds. Also, Lemma 2.4 leads to

$$\xi_1 - q\xi_2 < \frac{q+1}{p-q} < l. \quad (87)$$

Thus we see that if a triple $\{\eta_1, \eta_2, \eta_3\}$ leads to one of the cases (B) or (C), then inequality (56) holds. The inverse direction is true, because if inequality (56) holds, then one of the cases (B) or (C) is actual because:

1. If $m_a(1; \xi_1, \xi_2) > 0$, then inequality (56) leads to this conclusion.
2. If $m_a(1; \xi_1, \xi_2) \leq 0$, then the triple $\{\eta_1, \eta_2, \eta_3\}$ leads always to the case (B) or (C).

□

Lemma A.1. *If $\xi_1 > 1$ and $\xi_2 = 0$, then the function $g(a; \xi_1, 0)$ exhibits the behavior (C). Also, if $\xi_1 = 1$ and $\xi_2 = 0$, then the function $g(a; 1, 0)$ exhibits the behavior (B).*

Proof. Let us remind that the function $m(\cdot)$ is related to the derivative of $g(\cdot)$ through equation (53). If $\xi_1 > 1$ and $\xi_2 = 0$, then $m_a(0; \xi_1, 0) < 0$ and $m_a(1; \xi_1, 0) > 0$. Hence, the function $m(a; \xi_1, 0)$ is U -shaped since the derivative $m_a(a; \xi_1, 0)$, given by formula (80), is monotone. The inequalities $m(0; \xi_1, 0) > 0$ and $m(1; \xi_1, 0) > 0$ prove the first statement. The second one holds since the inequality $m(1; \xi_1, 0) > 0$ turns into equality when $\xi_1 = 1$. □

Lemma A.2. *Let $\bar{\xi}_2$ be defined by formula (57). Then the following inequality holds:*

$$\xi_1 - q\bar{\xi}_2 \geq 0. \quad (88)$$

Proof. Using Proposition 4.1, we derive

$$\eta_2 < \xi_2 < \frac{q^q}{(q+1)^{q+1}}. \quad (89)$$

Therefore,

$$\begin{aligned}
 \xi_1 - q\bar{\xi}_2 &= \eta_1 - \eta_2 - \left(\frac{q}{q+1}\right)^{q+1} \\
 &\geq \eta_1 - \frac{q^q}{(q+1)^{q+1}} - \left(\frac{q}{q+1}\right)^{q+1} \\
 &= \eta_1 - \left(\frac{q}{q+1}\right)^q \geq 0
 \end{aligned} \tag{90}$$

since $\eta_1 \geq 1$. □

Lemma A.3. *The function $f(a) = a^q - a^{q+1}$ achieves its maximum in the interval $(0, 1]$ for*

$$a = \frac{q}{q+1}. \tag{91}$$

Proof. The lemma holds since $f'(a) = a^{q-1}[q - a(q+1)]$. □

Proof of Proposition 4.9. The function $g(\bar{a}_1(\xi_1, 0); \xi_1, 0)$ is decreasing because, for $\xi_{1,1} < \xi_{1,2}$,

$$g(\bar{a}_1(\xi_{1,1}, 0); \xi_{1,1}, 0) > g(\bar{a}_1(\xi_{1,2}, 0); \xi_{1,1}, 0) > g(\bar{a}_1(\xi_{1,2}, 0); \xi_{1,2}, 0). \tag{92}$$

The first inequality holds since $\bar{a}(\xi_{1,1}, 0)$ maximizes $g(\cdot; \xi_{1,1}, 0)$. The second one is true because $g(a; \xi_1, 0)$ decreases w.r.t. ξ_1 when $a < 1$.

We have $g(\bar{a}_1(1, 0); 1, 0) > 0$ due to Lemma A.1 and the equality $g(1; 1, 0) = 0$. Let us consider the case $\xi_1 \rightarrow l$. Suppose that $\lim_{\xi_1 \rightarrow l} g(\bar{a}_1(\xi_1, 0); \xi_1, 0) \geq 0$. We have

$$\lim_{\xi_1 \rightarrow l} \bar{a}_1(\xi_1, 0) < \gamma, \tag{93}$$

where

$$\gamma = \left(\frac{q+1}{p-q}\right)^{\frac{1}{p-q-1}} \tag{94}$$

because γ is just the root of $m_a(a; \xi_1, 0) = 0$ when $\xi_1 \rightarrow l$. Having in mind that $\gamma < 1$ due to $r < 0$, we conclude that there exists some constant $b < \gamma$ such that $\lim_{\xi_1 \rightarrow l} g(b; \xi_1, 0) \geq 0$. On the other hand, this is impossible because the function $g(\cdot; l, 0)$ exhibits behavior (A) due to Proposition 4.7. □

Proof of Proposition 4.10. The function $g(\bar{a}_1(\xi_1, \xi_2); \xi_1, \xi_2)$ decreases w.r.t. ξ_2 because, for $\xi_{2,1} < \xi_{2,2}$,

$$g(\bar{a}_1(\xi_1, \xi_{2,1}); \xi_1, \xi_{2,1}) > g(\bar{a}_1(\xi_1, \xi_{2,2}); \xi_1, \xi_{2,1}) > g(\bar{a}_1(\xi_1, \xi_{2,2}); \xi_1, \xi_{2,2}). \tag{95}$$

The first inequality holds since $\bar{a}(\xi_1, \xi_{2,1})$ maximizes $g(\cdot; \xi_1, \xi_{2,1})$, whereas the second one is true because $g(a; \xi_1, \xi_2)$ decreases w.r.t. ξ_2 .

Also, $g(\bar{a}_1(\xi_1, 0); \xi_1, 0) > 0$ since the positive domain of the function $g(a; \xi_1, 0)$ is not empty.

Let us consider now the case $\xi_2 = \bar{\xi}_2$. Using the presentation

$$\xi_2 = \left(\frac{q}{q+1} \right)^q - \left(\frac{q}{q+1} \right)^{q+1} \quad (96)$$

and Lemmas A.2 and A.3, we derive

$$\begin{aligned} g(a; \xi_1, \bar{\xi}_2) &= a^p (\xi_1 - q\xi_2) + p(a^q - a^{q+1}) - (\xi_1 + \xi_2(p - q)) \\ &\leq a^p (\xi_1 - q\bar{\xi}_2) + p \left(\left(\frac{q}{q+1} \right)^q - \left(\frac{q}{q+1} \right)^{q+1} \right) - (\xi_1 + \bar{\xi}_2(p - q)) \\ &= a^p (\xi_1 - q\bar{\xi}_2) + \bar{\xi}_2 p - (\xi_1 + \bar{\xi}_2(p - q)) \\ &= -(1 - a^p) (\xi_1 - q\bar{\xi}_2) < 0. \end{aligned} \quad (97)$$

Particularly, $g(\bar{a}_1(\xi_1, \bar{\xi}_2); \xi_1, \bar{\xi}_2) < 0$. Hence, the value of ξ_2 that changes the sign of $g(\bar{a}_1(\xi_1, \xi_2); \xi_1, \xi_2)$ belongs to the interval $(0, \bar{\xi}_2)$. $\square$

Proof of Proposition 4.11. Note that the following relation holds:

$$h(a; \xi_2) = a^{p-q} g(a; \xi_1, \xi_2) + \frac{1 - a^p}{pa^{q-1}} g_a(a; \xi_1, \xi_2). \quad (98)$$

We know that the function $h(a; \xi_2)$ starts from a positive value and has only one root, see Lemma 3.4. Equation (98) taken in the points $a = \bar{a}_{1,2}(\xi_1, \xi_2)$ states that

$$\begin{aligned} h(\bar{a}_1(\xi_1, \xi_2); \xi_2) &> 0, \\ h(\bar{a}_2(\xi_1, \xi_2); \xi_2) &< 0 \end{aligned} \quad (99)$$

because $g_a(\bar{a}_{1,2}(\xi_1, \xi_2); \xi_1, \xi_2) = 0$ since both points are extrema of the function $g(a; \xi_1, \xi_2)$. Note that we need the inequality $h(1; \xi_2) = -p\xi_2 < 0$ if $\bar{a}_2(\xi_1, \xi_2) = 1$. Having in mind Lemma 3.4 and Proposition 4.3, we conclude

$$\bar{a}_1(\xi_1, \xi_2) < a^*(\xi_2) < \bar{a}_2(\xi_1, \xi_2). \quad (100)$$

Therefore $g_a(a^*(\xi_2); \xi_1, \xi_2) < 0$ because the function $g(a; \xi_1, \xi_2)$ decreases between the points $\bar{a}_1(\xi_1, \xi_2)$ and $\bar{a}_2(\xi_1, \xi_2)$. Thus equation (98) taken for $a = a^*(\xi_2)$ leads to $g(a^*(\xi_2); \xi_1, \xi_2) > 0$ since $h(a^*(\xi_2); \xi_2) = 0$ due to Proposition 4.3. $\square$

Proof of Lemma 4.15. Proposition 4.10 shows that $g(\bar{a}_1(x, \xi_2^*(x)); x, \xi_2^*(x)) = 0$ in the whole interval $(0, \bar{\xi}_2)$ and therefore

$$\frac{dg(\bar{a}_1(x, \xi_2^*(x)); x, \xi_2^*(x))}{dx} = 0. \quad (101)$$

This is equivalent to

$$(\xi_2^*(x))' = -\frac{g_{\xi_1}(\bar{a}_1(x, \xi_2^*(x)); x, \xi_2^*(x))}{g_{\xi_2}(\bar{a}_1(x, \xi_2^*(x)); x, \xi_2^*(x))}, \quad (102)$$

because

$$g_a(\bar{a}_1(x, \xi_2^*(x)); x, \xi_2^*(x)) = 0 \quad (103)$$

since $\bar{a}_1(x, \xi_2^*(x))$ is the maximum of the function $g(a; \cdot, \cdot)$. Thus

$$f'(x) = \frac{g_{\xi_1}(\bar{a}_1(x, \xi_2^*(x)); x, \xi_2^*(x))}{g_{\xi_2}(\bar{a}_1(x, \xi_2^*(x)); x, \xi_2^*(x))} - 1 < 0. \quad (104)$$

because $g(a; \cdot, \cdot)$ is decreasing both in ξ_1 and ξ_2 , and $g_{\xi_1}(\cdot, \cdot, \cdot) > g_{\xi_2}(\cdot, \cdot, \cdot)$ when $r < 0$. Therefore, $f(x)$ is a decreasing function. The inequality $f(0) > 0$ is obvious, whereas $f(\bar{\xi}_2) < 0$ due to Proposition 4.10. $\square$

Proof of Proposition 5.2. The derivative of the function $m(a; \xi_1, \xi_2)$,

$$m_a(a; \xi_1, \xi_2) = -a^{p-q-1}(p-q)(\xi_1 + q\xi_2) + q + 1, \quad (105)$$

is always negative because $m_a(1; \xi_1, \xi_2) < 0$ since $\xi_1 \geq 1$, $\xi_2 \geq 0$, and $p > 2q + 1$. Therefore, $m(a; \xi_1, \xi_2)$ is a decreasing negative function because $m(1; \xi_1, \xi_2) = -(\xi_1 - 1 + q\xi_2)$. Hence, the function $g(a; \xi_1, \xi_2)$ exhibits the behavior (A). $\square$

Proof of Proposition 5.3. The triple $\{\eta_1, \eta_2, \eta_3\}$ leads to one of the cases (B) or (C) if and only if $m_a(1; \xi_1, \xi_2) > 0$ and $m(\tilde{a}; \xi_1, \xi_2) > 0$, where $\tilde{a}$ is the larger than one root of $m_a(a; \xi_1, \xi_2) = 0$:

$$\tilde{a} = \left(\frac{q+1}{(p-q)(\xi_1 + q\xi_2)} \right)^{\frac{1}{p-q-1}}. \quad (106)$$

Using statement (23), we can check that the desired inequalities hold if and only if $\xi_1 + q\xi_2 < l$. $\square$

Acknowledgement

The author would like to express sincere gratitude to the editor Prof. Yuliya Mishura and to the anonymous reviewers for the helpful and constructive comments which substantially improve the quality of this paper.

Funding

This study is financed by the European Union-NextGenerationEU, through the National Recovery and Resilience Plan of the Republic of Bulgaria, project No BG-RRP-2.004-0008.

References

- Baurdoux, E.J., Kyprianou, A.E.: Further calculations for Israeli options. *Stoch. Int. J. Probab. Stoch. Process.* **76**(6), 549–569 (2004). [MR2100021. https://doi.org/10.1080/10451120412331313438](https://doi.org/10.1080/10451120412331313438)

- Bensoussan, A., Friedman, A.: Nonlinear variational inequalities and differential games with stopping times. *J. Funct. Anal.* **16**(3), 305–352 (1974). <https://www.sciencedirect.com/science/article/pii/0022123674900767>. ISSN 0022-1236. MR0354049. [https://doi.org/10.1016/0022-1236\(74\)90076-7](https://doi.org/10.1016/0022-1236(74)90076-7)
- Bensoussan, A., Friedman, A.: Nonzero-sum stochastic differential games with stopping times and free boundary problems. *Trans. Am. Math. Soc.* **231**(2), 275–327 (1977). <http://www.jstor.org/stable/1997905>. ISSN 0002-9947. MR0453082. <https://doi.org/10.2307/1997905>
- Black, F., Scholes, M.: The pricing of options and corporate liabilities. *J. Polit. Econ.* **81**(3), 637–654 (1973). MR3363443. <https://doi.org/10.1086/260062>
- Darling, D.A., Siegert, A.J.F.: The first passage problem for a continuous Markov process. *Ann. Math. Stat.* 624–639 (1953). MR0058908. <https://doi.org/10.1214/aoms/1177728918>
- Dolinsky, Y.: On shortfall risk minimization for game options. *Mod. Stoch. Theory Appl.* **7**(4), 379–394 (2020). ISSN 2351-6046. MR4195642. <https://doi.org/10.15559/20-VMSTA164>
- Dumitrescu, R., Quenez, M.-C., Sulem, A.: Game options in an imperfect market with default. *SIAM J. Financ. Math.* **8**(1), 532–559 (2017). MR3679314. <https://doi.org/10.1137/16M1109102>
- Dynkin, E.B.: A game-theoretic version of an optimal stopping problem. *Dokl. Akad. Nauk SSSR* **185**(1), 16–19 (1969). in Russian. MR0241121
- Ekström, E.: Properties of game options. *Math. Methods Oper. Res.* **63**(2), 221–238 (2006). ISSN 1432-5217. MR2264747. <https://doi.org/10.1007/s00186-005-0027-3>
- Ekström, E., Peskir, G.: Optimal stopping games for Markov processes. *SIAM J. Control Optim.* **47**(2), 684–702 (2008). MR2385859. <https://doi.org/10.1137/060673916>
- Ekström, E., Villeneuve, S.: On the value of optimal stopping games. *Ann. Appl. Probab.* **16**(3), 1576–1596 (2006). MR2260074. <https://doi.org/10.1214/105051606000000204>
- Emmerling, T.J.: Perpetual cancellable American call option. *Math. Finance* **22**(4), 645–666 (2012). MR2968279. <https://doi.org/10.1111/j.1467-9965.2011.00479.x>
- Friedman, A.: Stochastic games and variational inequalities. *Arch. Ration. Mech. Anal.* **51**(5), 321–346 (1973). MR0351571. <https://doi.org/10.1007/BF00263039>
- Gapeev, P.V., Lerche, H.R.: On the structure of discounted optimal stopping problems for one-dimensional diffusions. *Stoch. Int. J. Probab. Stoch. Process.* **83**(4–6), 537–554 (2011). MR2842594. <https://doi.org/10.1080/17442508.2010.532874>
- Gapeev, P.V.: The spread option optimal stopping game. In: *Exotic Option Pricing and Advanced Lévy Models*. pp. 293–305. (2005). MR2343219
- Gapeev, P.V., Li, L., Wu, Z.: Perpetual American cancellable standard options in models with last passage times. *Algorithms* **14**(1), (2021). <https://www.mdpi.com/1999-4893/14/1/3>. ISSN 1999-4893. MR4213424. <https://doi.org/10.3390/a14010003>
- Guo, I., Rutkowski, M.: Arbitrage-free pricing of multi-person game claims in discrete time. *Finance Stoch.* **21**(1), 111–155 (2017). MR3590704. <https://doi.org/10.1007/s00780-016-0315-1>
- Guo, P.: Pricing of the quanto game option with Asian feature. *J. Finance Account.* **8**(3), 143 (2020). <https://doi.org/10.11648/j.jfa.20200803.15>
- Guo, P., Chen, Q., Guo, X., Fang, Y.: Path-dependent game options: a lookback case. *Rev. Deriv. Res.* **17**(1), 113–124 (2014). <https://doi.org/10.1007/s11147-013-9092-6>
- Guo, P., Zhang, J., Wang, Q.: Path-dependent game options with Asian features. *Chaos Solitons Fractals* **141**, 110412 (2020). MR4171636. <https://doi.org/10.1016/j.chaos.2020.110412>
- Hamadène, S.: Mixed zero-sum stochastic differential game and American game options. *SIAM J. Control Optim.* **45**(2), 496–518 (2006). MR2246087. <https://doi.org/10.1137/S036301290444280X>

- Jacka, S.D.: Optimal stopping and the American put. *Math. Finance* **1**(2), 1–14 (1991). <http://dx.doi.org/10.1111/j.1467-9965.1991.tb00007.x>. ISSN 1467-9965. <https://doi.org/10.1111/j.1467-9965.1991.tb00007.x>
- Jacka, S.D.: Finite-horizon optimal stopping, obstacle problems and the shape of the continuation region. *Stoch. Int. J. Probab. Stoch. Process.* **39**(1), 25–42 (1992). [MR1293300. https://doi.org/10.1080/17442509208833761](https://doi.org/10.1080/17442509208833761)
- Kallsen, J., Kühn, C.: Pricing derivatives of American and game type in incomplete markets. *Finance Stoch.* **8**(2), 261–284 (2004). [MR2048831. https://doi.org/10.1007/s00780-003-0110-7](https://doi.org/10.1007/s00780-003-0110-7)
- Karatzas, I., Shreve, S.: *Methods of Mathematical Finance*. Springer, New York (1998). [MR1640352. https://doi.org/10.1007/b98840](https://doi.org/10.1007/b98840)
- Karatzas, I., Sudderth, W.: Stochastic games of control and stopping for a linear diffusion. In: *Random Walk, Sequential Analysis And Related Topics: A Festschrift in Honor of Yuan-Shih Chow*, pp. 100–117. World Scientific, (2006). [MR2367702. https://doi.org/10.1142/9789812772558_0007](https://doi.org/10.1142/9789812772558_0007)
- Kifer, Y.: Game options. *Finance Stoch.* **4**(4), 443–463 (Aug. 2000). ISSN 0949-2984. [MR1779588. https://doi.org/10.1007/PL00013527](https://doi.org/10.1007/PL00013527)
- Kifer, Y.: Dynkin's games and Israeli options. *ISRN Probability and Statistics* (2013, 2013)
- Kim, I.J.: The analytic valuation of American options. *Rev. Financ. Stud.* **3**(4), 547–572 (1990). ISSN 08939454, 14657368. <http://www.jstor.org/stable/2962115>. <https://doi.org/10.1093/rfs/3.4.547>
- Kühn, C., Kyprianou, A.E.: Callable puts as composite exotic options. *Math. Finance* **17**(4), 487–502 (2007). [MR2352903. https://doi.org/10.1111/j.1467-9965.2007.00313.x](https://doi.org/10.1111/j.1467-9965.2007.00313.x)
- Kühn, C., Kyprianou, A.E., Van Schaik, K.: Pricing Israeli options: a pathwise approach. *Stoch. Int. J. Probab. Stoch. Process.* **79**(1–2), 117–137 (2007). [MR2290401. https://doi.org/10.1080/17442500600976442](https://doi.org/10.1080/17442500600976442)
- Kunita, H., Seko, S.: Game call options and their exercise regions. Technical report, Nanzan Academic Society, Mathematical Sciences and Information Engineering, (2004).
- Kwok, Y.-K.: *Mathematical Models of Financial Derivatives*. Springer-Verlag, (2008). [MR2446710](https://doi.org/10.1007/978-3-540-70110-0)
- Kyprianou, A.E.: Some calculations for Israeli options. *Finance Stoch.* **8**(1), 73–86 (2004). [MR2022979. https://doi.org/10.1007/s00780-003-0104-5](https://doi.org/10.1007/s00780-003-0104-5)
- Lehoczky, J.P.: Formulas for stopped diffusion processes with stopping times based on the maximum. *Ann. Probab.* 601–607 (1977). [MR0458570. https://doi.org/10.1214/aop/1176995770](https://doi.org/10.1214/aop/1176995770)
- McKean, H.P.: A free boundary problem for the heat equation arising from a problem in mathematical economics. *Ind. Manag. Rev.* **6**(2), 32–39 (1965)
- Merton, R.C.: Theory of rational option pricing. *Bell J. Econ. Manag. Sci.* 141–183 (1973). [MR0496534. https://doi.org/10.2307/3003143](https://doi.org/10.2307/3003143)
- Palmowski, Z., Śtepiński, P.: Last-passage American cancelable option in Lévy models. *J. Risk Financ. Manag.* **16**(2), 82 (2023). <https://doi.org/10.3390/jrfm16020082>
- Peskir, G.: Optimal stopping games and Nash equilibrium. *Theory Probab. Appl.* **53**(3), 558–571 (2009). [MR2759714. https://doi.org/10.1137/S0040585X97983821](https://doi.org/10.1137/S0040585X97983821)
- Shiryaev, A.N., Kabanov, Y.M., Kramkov, D.O., Mel'nikov, A.V.: Toward the theory of pricing of options of both European and American types. II. Continuous time. *Theory Probab. Appl.* **39**(1), 61–102 (1995). [MR1348191. https://doi.org/10.1137/1139003](https://doi.org/10.1137/1139003)

- Suzuki, A., Sawaki, K.: The pricing of perpetual game put options and optimal boundaries. In: Recent Advances in Stochastic Operations Research, pp. 175–188. World Scientific, River Edge, NJ, USA (2007). [MR2313198](#). https://doi.org/10.1142/9789812706683_0012
- van Moerbeke, P.: On optimal stopping and free boundary problems. *Adv. Appl. Probab.* **5**(1), 33–35 (1973). [MR2940361](#). <https://doi.org/10.2307/1425961>
- Yam, S.C.P., Yung, S.P., Zhou, W.: Game call options revisited. *Math. Finance* **24**(1), 173–206 (2014). [MR3157693](#). <https://doi.org/10.1111/mafi.12000>
- Zaevski, T.: Perpetual cancellable American options with convertible features. *Mod. Stoch. Theory Appl.* **10**(4), 367–395 (2023). ISSN 2351-6046 (print), 2351-6054 (online). <https://www.vmsta.org/journal/VMSTA/article/273/read>. [MR4655406](#). <https://doi.org/10.15559/23-VMSTA230>
- Zaevski, T.: On the ϵ -optimality of American options. *China Finance Review International*, ahead-of-print, (2025). <https://doi.org/10.1108/CFRI-06-2024-0361>
- Zaevski, T.S.: Perpetual game options with a multiplied penalty. *Commun. Nonlinear Sci. Numer. Simul.* **85**, 105248 (2020). ISSN 1007-5704 (print), 1878-7274 (online). <http://www.sciencedirect.com/science/article/pii/S1007570420300812>. [MR4074142](#). <https://doi.org/10.1016/j.cnsns.2020.105248>
- Zaevski, T.S.: Discounted perpetual game call options. *Chaos Solitons Fractals* **131**, 109503 (2020). ISSN 0960-0779 (print), 1873-2887 (online). <http://www.sciencedirect.com/science/article/pii/S0960077919304552>. [MR4065328](#). <https://doi.org/10.1016/j.chaos.2019.109503>
- Zaevski, T.S.: Discounted perpetual game put options. *Chaos Solitons Fractals* **137**, 109858 (2020). ISSN 0960-0779 (print), 1873-2887 (online). <http://www.sciencedirect.com/science/article/pii/S0960077920302587>. [MR4099368](#). <https://doi.org/10.1016/j.chaos.2020.109858>
- Zaevski, T.S.: A new approach for pricing discounted American options. *Commun. Nonlinear Sci. Numer. Simul.* **97**, 105752 (2021). ISSN 1007-5704 (print), 1878-7274 (online). <https://www.sciencedirect.com/science/article/pii/S1007570421000630>. [MR4212896](#). <https://doi.org/10.1016/j.cnsns.2021.105752>
- Zaevski, T.S.: Pricing finite maturity game call options with convertible features. *Commun. Stat., Simul. Comput.* 1–24 (2025). <https://doi.org/10.1080/03610918.2025.2488972>